CRAMER-RAO BOUNDS AND PARAMETER ESTIMATION FOR RANDOM AMPLITUDE PHASE MODULATED SIGNALS

Mounir Ghogho, Asoke K. Nandi *
University of Strathclyde
Dept of E.E.E.
204, George Street
G1 1XW, United Kingdom
mounir@spd.eee.strath.ac.uk, asoke@eee.strath.ac.uk

Ananthram Swami
Army Research Lab
Attn: AMSRL-IS-TA
2800 Powder Mill Road
Adelphi, MD 20783-1197, USA
e-mail: a.swami@ieee.org

ABSTRACT

The problem of estimating the phase parameters of a phase modulated signal in the presence of coloured multiplicative noise (random amplitude modulation) and additive white noise, both Gaussian, is addressed. Closed-form expressions for the exact and large-sample Cramer-Rao Bounds (CRB) are derived. It is shown that the CRB is not significantly affected by the colour of the modulating process, especially when the signal-to-noise ratio is high. Hence, maximum likelihood type estimators which ignore the noise colour and optimize a criterion with respect to only the phase parameters are proposed. These estimators are shown to be equivalent to the nonlinear least squares estimators which consist of matching the squared observations with a constant amplitude phase modulated signal when the mean of the multiplicative noise is forced to zero. Closed-form expressions are derived for the efficiency of these estimators, and are verified via simulations.

1. INTRODUCTION

The estimation of the instantaneous frequency of Phase Modulated Signals (PMS) is a problem which arises in many applications. For example, in coherent radar systems, the phase modulation is directly related to the target range [7, ch 7, p58-65]. The important case of harmonic signals is obtained when the radial velocity of the target is constant. The phase variations can often be approximated by a finite-order (often low) polynomial, and the resulting signal model is called the Polynomial Phase Signal (PPS) [6]. The estimation of the phase parameters when the amplitude signal is constant has received much attention in the literature. Recent papers have addressed the more general problem where the signal amplitude is randomly time-varying; see, e.g., [5, 8, 3]. Random amplitude modulation, or multiplicative noise, shows up, for example, in active sonar due to dispersion in the medium [1]. The signal model considered here is then:

$$x(t) = s(t)e^{j\phi(t)} + v(t), \qquad t = 0, ..., N-1, \tag{1}$$

where $s(t)$ and $\phi(t)$ are the instantaneous amplitude and phase, respectively, and $v(t)$ is additive noise. We make the following assumptions:
(AS1) The amplitude signal $s(t)$ is Gaussian and real-valued, but not necessarily stationary
(AS2) $v(t)$ is zero-mean circular white Gaussian with variance σ_v^2.
(AS3) $\phi(t)$ is a deterministic function of time which is parameterized by a finite dimensional parameter vector, $\boldsymbol{\varphi} = [\varphi_0, ..., \varphi_M]^T$; $\phi(t)$ is differentiable with respect to $\boldsymbol{\varphi}$.

*This work was partly supported by the Engineering and Physical Sciences Research Council of UK under research grant ref. GR/L07475.

Since the mean of $s(t)$ is allowed to be time-varying, PMS with deterministic amplitude are seen to be special cases of (1). Thus, the model (1) under (AS1)-(AS3) is quite general. In later sections, we will constrain (AS1) by assuming that the autocovariance function of $s(t)$ is time-invariant. This will be referred to as assumption (AS1')

To assess the performance of the estimators of the phase parameters in (1), we derive the Cramer-Rao Bound (CRB). The CRB for a constant amplitude PPS in circular white Gaussian noise was derived in [6]. The CRB for model (1) under assumptions (AS1)-(AS3) has been considered in [2] under additional assumptions (the covariance function was assumed to be stationary, and both $\mu_s(t)$ and $\phi(t)$ were expressed as linear combinations of known basis functions; neither assumption is made in this paper). The real-valued case was studied in [9]. Here, we derive expressions for the CRB which are: *i*) computationally attractive; *ii*) facilitate the derivation of the asymptotic (large-sample) CRB; *iii*) lead to new insights into the behavior of the error bounds.

The second part of the paper is devoted to the estimation of the phase parameters in (1). To reduce the complexity of the exact Maximum Likelihood (ML) estimators, we ignore the colour of the amplitude $s(t)$. When the mean of $s(t)$ is forced to zero in this pseudo ML scheme, the resulting estimator is shown to coincide with the nonlinear least squares estimators which consist of matching the squared observation with a constant amplitude PMS. The degradations introduced by the zero-mean and i.i.d. assumptions are studied analytically and through simulations. It is shown that these degradations are not very significant, especially when the Signal-to-Noise Ratio (SNR) is high.

2. AN EQUIVALENT MODEL

Since $v(t)$ is zero-mean, white, circular and Gaussian, $v(t)$ and $v(t)e^{j\phi(t)}$ are statistically equivalent. Indeed, $v(t)e^{j\phi(t)}$ is a zero-mean Gaussian variable; hence, it is completely characterized by

$$r = \mathrm{E}\left\{\left|v(t)e^{j\phi(t)}\right|^2\right\} \quad \text{and} \quad c = \mathrm{E}\left\{\left(v(t)e^{j\phi(t)}\right)^2\right\}$$

where E denotes the mathematical expectation operator. Parameter r is the variance, σ_v^2, and the circularity of $v(t)$ implies that $c = 0$. A statistically equivalent model for (1) is then given by

$$x(t) = (s(t) + v(t))e^{j\phi(t)} \; . \tag{2}$$

In fact, this equivalence holds for any $v(t)$ which is iid and circularly symmetric, and not necessarily Gaussian.

3. CRAMER-RAO BOUND

Let $\mu_s(t)$ and $r_s(t, \tau)$ denote the mean and the covariance function of the amplitude process. We assume that

$\mu_s(t)$ and $r_s(t,\tau)$ are described by a finite dimensional non-random parameter vector $\boldsymbol{\alpha} = [\alpha_1, ..., \alpha_p]^T$. The complete deterministic parameter vector of the signal model in (2) is $\theta = [\boldsymbol{\varphi}^T, \boldsymbol{\alpha}^T, \sigma_v^2]^T$. Denote the observed data vector by

$$\mathbf{x} \;=\; [x(0), ..., x(N-1)]^T \;. \qquad (3)$$

Let $\mathbf{x}_r = \mathcal{R}\{\mathbf{x}\}$ and $\mathbf{x}_i = \mathcal{I}\{\mathbf{x}\}$, where $\mathcal{R}$ and $\mathcal{I}$ denote the real and imaginary parts. The elements of these vectors are given by [using model (2)]

$$x_r(t) \;=\; (s(t) + v_r(t))\cos\phi(t) - v_i(t)\sin\phi(t) \qquad (4)$$
$$x_i(t) \;=\; (s(t) + v_r(t))\sin\phi(t) + v_i(t)\cos\phi(t) \qquad (5)$$

where $v_r(t)$ and $v_i(t)$ are the real and imaginary parts of $v(t)$. Let $\mathbf{s}$, $\mathbf{v}_r$ and $\mathbf{v}_i$ be defined similar to $\mathbf{x}$ in (3), and let $\widetilde{\mathbf{x}} = [\mathbf{x}_r^T, \mathbf{x}_i^T]^T$. Let $\boldsymbol{\mu}_s$ and R_s denote the mean vector and covariance matrix of $\mathbf{s}$. Notice that $\mathbf{v}_r$ and $\mathbf{v}_i$ are i.i.d. sequences of zero-mean Gaussian variables with variance $\sigma_v^2/2$, and are mutually independent. We will find it useful to define the partial derivatives, $\phi_k(t) = \partial\phi(t)/\partial\varphi_k$.

3.1. Likelihood Function

Consider the change of variables $(\mathbf{s} + \mathbf{v}_r, \mathbf{v}_i) \rightarrow (\mathbf{x}_r, \mathbf{x}_i)$, where the elements of $\mathbf{x}_r$ and $\mathbf{x}_i$ are given in eqs. (4) and (5). The Jacobian of this transformation is unity. The vectors $(\mathbf{s} + \mathbf{v}_r)$ and $\mathbf{v}_i$ can be expressed as

$$\mathbf{s} + \mathbf{v}_r = \mathcal{R}\{\mathbf{x} \odot e^{-j\phi(\mathbf{t})}\}, \quad \mathbf{v}_i = \mathcal{I}\{\mathbf{x} \odot e^{-j\phi(\mathbf{t})}\} \qquad (6)$$

where $e^{-j\phi(\mathbf{t})} = [e^{-j\phi(0)}, ..., e^{-j\phi(N-1)}]^T$, and $\odot$ denotes element-wise multiplication (the Hadamard product). Since $\mathbf{s}$ and $\mathbf{v}_r$ are Gaussian and mutually independent, the vector $(\mathbf{s} + \mathbf{v}_r)$ is Gaussian with mean vector $\boldsymbol{\mu}_s$ and covariance matrix $R = R_s + I\,\sigma_v^2/2$, where I denotes the $(N \times N)$ identity matrix. Since $\mathbf{v}_i$ is independent of $(\mathbf{s} + \mathbf{v}_r)$, the log-likelihood function (LLF) of $\widetilde{\mathbf{x}}$ for a given θ is, after dropping constant terms, found to be

$$\begin{aligned} &\ln L(\widetilde{\mathbf{x}}/\theta) = -\tfrac{1}{2}\ln\det(R) - \tfrac{N}{2}\ln\sigma_v^2 \\ &-\tfrac{1}{2}\Big[\mathcal{R}\{\mathbf{x}\odot e^{-j\phi(\mathbf{t})}\} - \boldsymbol{\mu}_s\Big]^T R^{-1}\Big[\mathcal{R}\{\mathbf{x}\odot e^{-j\phi(\mathbf{t})}\} - \boldsymbol{\mu}_s\Big] \qquad (7) \\ &-\tfrac{1}{\sigma_v^2}\Big[\mathcal{I}\{\mathbf{x}\odot e^{-j\phi(\mathbf{t})}\}\Big]^T\Big[\mathcal{I}\{\mathbf{x}\odot e^{-j\phi(\mathbf{t})}\}\Big] \;. \end{aligned}$$

3.2. Fisher Information Matrix

Under our modeling assumptions, we obtain a closed-form expression for the Fisher Information Matrix (FIM); recall that the CRB is the inverse of the FIM.

Proposition 1. Under assumptions (AS1)-(AS3), the FIM is block diagonal, i.e., $J_{\boldsymbol{\varphi},\boldsymbol{\alpha}} = \mathbf{0}$ and $J_{\boldsymbol{\varphi},\sigma_v^2} = \mathbf{0}$, and the CRB for the phase parameter vector is given by

$$CRB(\boldsymbol{\varphi}) = J^{-1}_{\boldsymbol{\varphi},\boldsymbol{\varphi}}$$

where, for $k, \ell = 0, ..., M$, and

$$J_{\varphi_k,\varphi_\ell} = \sum_{t=0}^{N-1}\phi_k(t)\phi_\ell(t)\left[2\frac{\mathrm{E}\{s^2(t)\}}{\sigma_v^2} + \frac{\sigma_v^2}{2}\gamma(t) - 1\right] \qquad (8)$$

where $\gamma(t)$ is the t-th element of the diagonal of R^{-1}.
Proof: See [4]. □

We make the following interesting observations:

- Since $J_{\theta,\theta}$ is block-diagonal, the CRB for the phase parameters are the same whether or not the noise parameters $(\sigma_v^2, \boldsymbol{\mu}_s, R_s)$ are known. This decoupling was also noted in [2] but under additional assumptions.
- When $\phi(t)$ is a multi-linear function of $\boldsymbol{\varphi}$, the CRBs for the phase parameters are independent of the actual values of these parameters. This is in contrast with the real-valued case (see e.g., [9]) where this independence holds true only asymptotically, i.e., for large values of N. Furthermore, this property cannot be inferred directly from the FIM formula given in [2, eq. 81], which explicitly involves the phase waveform (recall that [2] makes additional restrictive assumptions). For the PPS $\phi(t) = \sum_{k=0}^{M}\varphi_k t^k$, $\phi_k(t) = t^k$ in (8).
- When the covariance function of $s(t)$ is time-invariant, matrix R_s is Toeplitz and the $\gamma(t)$'s can be computed recursively using the Levinson-Durbin algorithm: Let $\mathbf{g}$ denote the length $N-1$ linear prediction filter corresponding to the $N \times N$ autocorrelation matrix R, i.e., $R\mathbf{g} = \sigma_u^2[1, \mathbf{0}]'$, with $g(0) = 1$. Then,
$$\gamma(t) = \gamma(t-1) + \frac{|g(i)|^2 - |g(N-i)|^2}{\gamma(0)}\;, \;\; i = 1, ..., N-1,$$
A closed-form expression for the large sample case is given in the next section.
- If the SNR is low, i.e., σ_v^2 is large, then, $R \approx 0.5\sigma_v^2 I$ and $\gamma(t) = 2/\sigma_v^2\ \forall t$.
- If the SNR is high, i.e., σ_v^2 is small, $R \approx R_s$. If $s(t)$ is an AR(p) process, we can write $\gamma(t)$ explicitly in terms of the AR parameters, $\gamma(t) = \gamma(0)\sum_{k=0}^{p}|a(k)|^2$, $p < t < N-p$, so that if $N >> p$, the end-effects are small, and we may assume that $\gamma(t)$ is a constant. These end effects are negligible for PPS-type signals, even for finite N, since the relevant FIMs involve sums of the form $\gamma(t)t^k$.
- When the signal amplitude is temporally independent, i.e., $r_s(t,\tau) = \sigma_s^2(t)\delta(\tau)$, and $R_s = \mathrm{diag}(\sigma_s^2(t))$, the FIM entries $J_{\varphi_k,\varphi_\ell}$ are given by
$$J_{\varphi_k,\varphi_\ell} = \sum_{t=0}^{N-1}\phi_k(t)\phi_\ell(t)\left[2\frac{\mu_s^2(t)}{\sigma_v^2} + \frac{4\sigma_s^4(t)/\sigma_v^4}{2\sigma_s^2(t)/\sigma_v^2 + 1}\right]$$
- When the signal amplitude is deterministic, i.e., $\boldsymbol{\mu}_s = \mathbf{s}$ and $R_s = \mathbf{0}$, we obtain
$$J_{\varphi_k,\varphi_\ell} = \sum_{t=0}^{N-1}\phi_k(t)\phi_\ell(t)\,\left[2s^2(t)/\sigma_v^2\right].$$

4. ASYMPTOTIC CRB

In this section, we assume that the autocovariance function, but not the mean, of the amplitude signal is time-invariant, i.e., $r_s(t,\tau) \equiv r_s(\tau)$, $\forall t$, i.e., assumption (AS1') is in force. We make the mild assumption that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\mu_s^2(t) = \mu_{so}^2 \qquad (9)$$

where μ_{so}^2 is finite. We also assume that there exist real numbers $n(k)$ and $n(\ell)$ such that, for k, $\ell = 0, ..., M$,

$$a_N(k,\ell) \triangleq \frac{1}{N}\sum_{t=0}^{N-1}\frac{\phi_k(t)\phi_\ell(t)}{N^{n(k)+n(\ell)}} \underset{N\to\infty}{\longrightarrow} a(k,\ell) \qquad (10)$$

where the $a(k,\ell)$'s are finite constants, and the matrix $A = \{a(k,\ell)\}$ is non-singular. Also define

$$b_N(k,\ell) \;=\; \frac{1}{N}\sum_{t=0}^{N-1}\frac{\phi_k(t)\phi_\ell(t)}{N^{n(k)+n(\ell)}}\mu_s^2(t)$$

$$c_N(k,\ell) \;=\; \frac{1}{N}\sum_{t=0}^{N-1}\frac{\phi_k(t)\phi_\ell(t)}{N^{n(k)+n(\ell)}}\gamma(t)$$

Following Proposition 1, we consider

$$N^{n(k)+n(\ell)+1}J_{\varphi_k,\varphi_\ell} = \left(\frac{2\sigma_s^2}{\sigma_v^2}-1\right)a_N(k,\ell)$$
$$+\frac{2}{\sigma_v^2}b_N(k,\ell)+\frac{\sigma_v^2}{2}\ .c_N(k,\ell) \qquad (11)$$

We assume that the limit of $b_N(k,\ell)$, $k,\ell=0,...,M$, as N tends to infinity exists and is given by

$$b_N(k,\ell) \underset{N\to\infty}{\longrightarrow} \mu_{so}^2\, a(k,\ell) \qquad (12)$$

where μ_{so}^2 is given in (9). This condition is fulfilled, for example, when $\mu_s(t)$ is time-invariant or periodically time-varying.

Recall that in the case of finite N, the diagonal of the inverse of a Toeplitz matrix is not constant, so that R^{-1} is not the covariance matrix of a stationary process. As $N\to\infty$, R^{-1} also becomes Toeplitz, and the asymptotic values of the (constant) diagonal elements of R^{-1} are

$$\gamma(t) \underset{N\to\infty}{\longrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{S_s(\omega)+\sigma_v^2/2} \triangleq \gamma, \qquad \forall t \qquad (13)$$

where $S_s(\omega)$ is the power spectrum of $s(t)$; hence,

$$c_N(k,\ell) \underset{N\to\infty}{\longrightarrow} \gamma\ a(k,\ell)$$

Therefore, we obtain

$$N^{n(k)+n(\ell)+1}J_{\varphi_k,\varphi_\ell} \underset{N\to\infty}{\longrightarrow} \eta\ a(k,\ell), \qquad k,\ell=0,...,M$$

with

$$\eta \triangleq 2SNR+\frac{\sigma_v^2}{2}\gamma-1 \qquad (14)$$

where we have defined the SNR as $SNR \triangleq (\mu_{so}^2+\sigma_s^2)/\sigma_v^2$. The large sample FIM block $J_{\boldsymbol{\varphi},\boldsymbol{\varphi}}$ can be written as

$$J_{\boldsymbol{\varphi},\boldsymbol{\varphi}} \underset{N\to\infty}{\approx} \eta N\,\Psi\,A\,\Psi \qquad (15)$$

where $\Psi = diag\{N^{n(0)},...,N^{n(M)}\}$ and $A=\{a(k,\ell\}_{k,\ell=0}^M\}$.

Proposition 2. Under assumptions (AS1'), (AS2), (AS3), (9), (10) and (12), the large sample CRB for the φ_m's are

$$ACRB(\varphi_m)=\eta^{-1}\frac{A_{m,m}^{-1}}{N^{2n(m)+1}}, \qquad m=0,...,M \qquad (16)$$

where $A_{m,m}^{-1}$ denotes the mth diagonal element of A^{-1}. □

Thus, an efficient estimator can be consistent only if $n(m) > -0.5$. The matrix A depends only upon the parametrization of the phase function; the influence of the coloured amplitude modulation and the additive white noise are completely captured by η which can also be written as

$$\eta = 2\frac{\mu_{so}^2+\sigma_s^2}{\sigma_v^2}-1+\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{S_s(\omega)}{\sigma_v^2/2}+1\right]^{-1}d\omega\ .$$

Parameter η consists of two terms: a time-averaged SNR and a frequency-averaged inverse SNR. In the next section, we study the relationship between η and the bandwidth of $S_s(\omega)$. Parameter η can be interpreted as an effective SNR, and reduces to the usual definition of SNR for the constant amplitude case.

In the important case of PPS, A is the Hilbert matrix whose (k,ℓ) element is $1/(k+\ell+1)$, $k,\ell=0,...M$, $n(m)=m$, and $\Psi=diag\{1,N,...,N^M\}$ [6]; an expression for A^{-1} is given in [6]. As in the case of the constant amplitude PPS in additive noise, the large sample CRB for φ_m is of order $1/N^{2m+1}$, even when the amplitude is a random process.

5. EFFECT OF THE SIGNAL AMPLITUDE BANDWIDTH

We consider the asymptotic CRB given in Proposition 2. To get insight into the influence of the spectrum of the random amplitudes on the CRB, consider a band-limited process:

$$S_s(\omega)=\begin{cases}\frac{\sigma_s^2}{2B} & \text{if } |\omega-\omega_o|\le 2\pi B\\ 0 & \text{if } |\omega-\omega_o|>2\pi B\end{cases} \qquad (17)$$

where ω_o is the center frequency of $S_s(\omega)$ and $0\le B\le 0.5$. Substituting (17) into (13) and using (14), we find

$$\eta^{-1}\triangleq\frac{R_2+B}{2R_2(R_1+R_2)+2R_1B}$$

where $R_1=\mu_{so}^2/\sigma_v^2$ and $R_2=\sigma_s^2/\sigma_v^2$ denote the coherent and non coherent SNRs of the PMS. Parameter η^{-1} is an increasing function of the bandwidth parameter B. This indicates that the CRBs for the phase parameters increase with the bandwidth of the amplitude signal $s(t)$; indeed, as the bandwidth of $s(t)$ increases, we have more smearing, making parameter estimation harder. To quantify the effect of the signal amplitude bandwidth on the asymptotic CRB, we consider

$$\xi\triangleq\frac{\max_B ACRB(\varphi_k)}{\min_B ACRB(\varphi_k)}=1+\frac{1}{2SNR+ISNR}$$

where we have defined the intrinsic SNR as $ISNR\triangleq\mu_{so}^2/\sigma_s^2=R_1/R_2$. Notice that ξ is the same for all the φ_k's, $k=0,...,M$, i.e., it is independent of the parametrization, and ξ is almost unity unless both SNR and $ISNR$ are very low. Therefore, for high SNR (i.e., R_1 and/or R_2 large), the influence of the bandwidth of the signal amplitude is not significant and η^{-1} is approximated by

$$\eta^{-1}\approx\left(\eta|_{B=0}\right)^{-1}=\frac{1}{2\,SNR}=\frac{\sigma_v^2}{2(\mu_{so}^2+\sigma_v^2)} \qquad (18)$$

When $B=0$, the amplitude process is either a constant or a harmonic. This approximation also holds true when $ISNR$ is high regardless of the value of SNR. Indeed, in this case, the amplitude signal is mainly deterministic and the effects of its random component are minor. On the other hand, if $ISNR$ is low, the deviation of ξ from 1 may be large for small values of SNR since

$$\xi|_{ISNR=0}-1=\frac{1}{2SNR}=\frac{1}{2R_2}=\frac{\sigma_v^2}{2\sigma_s^2}\ .$$

6. MAXIMUM LIKELIHOOD ESTIMATION

According to section 5, the colour of the random amplitude modulation does not significantly affect the CRB when the SNR is high, regardless of the value of the intrinsic SNR. Motivated by this result, we ignore the colour of $s(t)$ and we force its mean to be zero, i.e., $\mu_s(t)=0$, $r_s(t,\tau)=\sigma_s^2\delta(\tau)$, $\forall t$, for the estimation of the phase parameters. Making this assumption reduces the complexity of the exact ML estimate. In section 7, we assess the asymptotic performance of these pseudo ML estimators and the degradations introduced by these assumptions.

Instead of σ_s^2 we consider $\sigma^2=\sigma_s^2+\sigma_v^2/2$; σ^2 and σ_v^2 may be viewed as independent parameters since $\sigma_s^2\neq 0$. The parameter vector is $\theta=[\boldsymbol{\varphi}^T,\sigma^2,\sigma_v^2]^T$, and the LLF in (7) reduces to

$$\ln L(\tilde{\mathbf{x}}/\theta) = -\frac{N}{2}\ln(\sigma^2)-\frac{1}{2\sigma^2}\sum_{t=0}^{N-1}\left[\mathcal{R}\{x(t)e^{-j\phi(t)}\}\right]^2$$
$$-\frac{N}{2}\ln\sigma_v^2-\frac{1}{\sigma_v^2}\sum_{t=0}^{N-1}\left[\mathcal{I}\{x(t)e^{-j\phi(t)}\}\right]^2 \qquad (19)$$

If $\widehat{\boldsymbol{\varphi}}$ denotes the ML estimate of $\boldsymbol{\varphi}$, the ML estimates of σ^2 and σ_v^2 are given by

$$\widehat{\sigma}^2 = \frac{1}{N}\sum_{t=0}^{N-1}\left[\mathcal{R}\{x(t)e^{-j\widehat{\phi}(t)}\}\right]^2 \qquad (20)$$

$$\widehat{\sigma}_v^2 = \frac{2}{N}\sum_{t=0}^{N-1}\left[\mathcal{I}\{x(t)e^{-j\widehat{\phi}(t)}\}\right]^2 \qquad (21)$$

where $\widehat{\phi}(t)$ is the phase sequence when $\boldsymbol{\varphi} = \widehat{\boldsymbol{\varphi}}$. Replacing σ^2 and σ_v^2 by their ML estimates in (19), and dropping constant terms, we see that the ML estimate of $\boldsymbol{\varphi}$ maximizes

$$-\frac{1}{N^2}\sum_{t=0}^{N-1}\left[\mathcal{R}\{x(t)e^{-j\phi(t)}\}\right]^2\sum_{t=0}^{N-1}\left[\mathcal{I}\{x(t)e^{-j\phi(t)}\}\right]^2$$

or equivalently

$$C(\boldsymbol{\varphi}) = \left[\frac{1}{N}\sum_{t=0}^{N-1}\mathcal{R}\{x^2(t)e^{-2j\phi(t)}\}\right]^2. \qquad (22)$$

The ML estimates of the phase parameters are then

$$(\widehat{\varphi}_0, ..., \widehat{\varphi}_M) = \arg\max_{\boldsymbol{\varphi}} C(\boldsymbol{\varphi}) \qquad (23)$$

Note that (22) is equivalent to a non-linear least squares algorithm which matches the squared data to $\exp(j2\phi(t))$ under the assumption that the amplitude is constant (see [4]). Since (23) is a nonlinear optimization problem, we must resort to numerical optimization techniques. In the case of the harmonic signal, (i.e., $\phi(t) = \varphi_1 t + \varphi_0$), the ML estimate of the frequency φ_1 is obtained by peak picking the discrete Fourier transform of the squared data. This is equivalent to the Cyclic-Variance-based estimator in [5, 3].

7. PERFORMANCE ANALYSIS

In this section, we derive expressions for the asymptotic performance of the phase parameter estimator in (23).

Proposition 3. Under assumptions (AS1'), (AS2), (AS3), (9), (10) and (12), the asymptotic variances of the phase parameter estimates in (23) are given by

$$\text{var}(\widehat{\varphi}_m) = \frac{1}{2}\frac{SNR+0.5}{SNR^2}\frac{A^{-1}_{m,m}}{N^{2n(m)+1}}, \qquad m = 0, ..., M$$

Proof: see [4]. □

Using Propositions 2 and 3, the asymptotic relative efficiency (ARE) of the estimate in (23) is thus

$$ARE \triangleq \frac{ACRB(\widehat{\varphi}_m)}{\text{var}(\widehat{\varphi}_m)} = \frac{2\eta^{-1}SNR^2}{SNR+0.5}$$

For the bandlimited model in (17), the ARE is

$$ARE = \frac{SNR}{SNR+0.5}\left[1 + \frac{B}{SNR + B \times ISNR}\right]$$

The ARE is an increasing function of B; hence

$$ARE \geq ARE|_{B=0} = SNR/(SNR+0.5) \qquad (24)$$

Moreover, this minimal value increases with SNR. For instance, $ARE > 0.8$ when $SNR > 2$ and $ARE > 0.95$ when $SNR > 10$. Thus, at high SNR, $ARE \approx 1$, and the estimate in (23) is almost asymptotically efficient. Notice that when $B = 0.5$ and $ISNR = 0$, $ARE = 1$, as expected, since the estimates in (23) are ML in this case.

8. SIMULATION RESULTS

We study the performance of the pseudo ML estimator in (23) through Monte-Carlo simulations. The amplitude signal $s(t)$ is a zero-mean stationary AR(1) process with regression parameter a. We keep its power σ_s^2 fixed and vary a, which controls the bandwidth of $s(t)$. Here, we limit the simulations to the harmonic signal case, i.e., $\phi(t) = \varphi_1 t + \varphi_0$; additional simulation results are reported in [4]. The parameters were set to $\varphi_1 = 0.4\pi$ and $\varphi_0 = 0.2\pi$. Table 1 displays the Relative Efficiency (RE) of the frequency estimator in (23) for different values of a and N when $SNR = 2$. The MSEs of the estimators were evaluated using 1000 Monte-Carlo simulations. In the computation of the RE, we used the exact CRB derived in Proposition 1. The last column of Table 1 displays the ARE computed using Propositions 2 and 3. For an AR(1) process with parameter a, η in eq. (14) is given by [4]

$$\eta = 2SNR - \frac{2R_2(1-a^2)}{\sqrt{[2R_2(1-a^2)+(1+a^2)]^2 - 4a^2}}$$

which we used to compute the ACRB via (16). It is seen that the theoretical expectations are met. In particular, the RE is seen to exceed 0.8 for large number of samples, as predicted by eq. (24).

	$N = 100$	$N = 200$	$N = 1000$	ARE
$a = -0.9$	0.50	0.70	0.89	0.90
$a = -0.5$	0.76	0.88	0.97	0.97
$a = 0$	0.89	0.95	1.00	1.00
$a = 0.5$	0.88	0.91	0.97	0.97
$a = 0.9$	0.22	0.66	0.88	0.90

Table 1. $RE(\widehat{\varphi}_1) = CRB(\varphi_1)/MSE(\widehat{\varphi}_1)$

REFERENCES

[1] R. F. Dwyer, "Fourth-order spectra of a Gaussian amplitude-modulated sinusoids," *J. Acoust. Soc. Amer.*, Aug. 1991.

[2] J. Francos and B. Friedlander, "Bounds for estimation of multicomponent signals with random amplitude and deterministic phase," *IEEE Trans. Sig. Proc.*, May 1995.

[3] M. Ghogho, A. Swami and A. K. Nandi, "Nonlinear least squares estimation for harmonics in multiplicative and additive noise," *Proc. SSAP'98*, Portland (OR), Sept. 1998.

[4] M. Ghogho, A. K. Nandi and A. Swami, "Cramer-Rao bounds and maximum likelihood estimation for random amplitude modulated phase signals," *Technical Report, EEE Dept., Univ. Strathclyde, Sept 1998.*

[5] G.B. Giannakis and G. Zhou, "Harmonics in multiplicative and additive noise: parameter estimation using cyclic statistics," *IEEE Trans. Sig. Proc.*, Sept. 1995.

[6] S. Peleg and B. Porat, "The Cramer-Rao lower bound for signals with constant amplitude and polynomial phase," *IEEE Trans. Sig. Proc.*, March. 1991.

[7] A.W. Rihaczek, *Principles of high-resolution radar*, Los Altos, CA: Peninsula, 1985.

[8] S. Shamsunder, G.B. Giannakis and B. Friedlander, "Estimating random amplitude polynomial phase signals: a cyclostationary approach," *IEEE Trans. Sig. Proc.*, Feb. 1995.

[9] A. Swami, "Cramer-Rao bounds for deterministic signals in additive and multiplicative noise," *Signal Processing*, Aug. 1996.