

CLOSED-FORM AND REAL-TIME WORDLENGTH ADAPTATION

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ABSTRACT

FPGA and configurable computing-based DSP algorithms have demonstrated significant performance improvements over software implementations. This has caused recent renewed interest in developing or mapping DSP algorithms to custom hardware. An algorithm will be successfully mapped if the intermediate wordlengths can be reduced to maintain reasonable hardware size. In this paper, we consider linear hardware cost functions, for which we can derive closed-form expressions for the reduced wordlengths. We then apply these results to an adaptive LMS filter, where we adapt not only the tap weights, but also the wordlengths as a function of the data in real-time.

1. INTRODUCTION

Recent advances in field programmable gate array (FPGA) [1, 2] and configurable computing technology [3, 4] have caused renewed interest in custom digital signal processing (DSP) hardware designs. For many DSP problems, reduced precision arithmetic will maintain acceptable system performance. FPGA-based DSP computers can be programmed to bit-level granularity, allowing for a more efficient implementation than is possible for a conventional programmable computer [5]. A mapping of an algorithm to an FPGA architecture will be successful if the designer can limit wordlength growth without sacrificing algorithm performance.

In order to reduce hardware requirements, one must round or truncate data at various points throughout an algorithm. Many operations such as multiplication or accumulation can greatly expand the wordlength of the data. If these operations are cascaded, then without some form of wordlength reduction, excessive hardware growth results. Wordlength reduction introduces noise into the data stream, so the designer must balance the need for an efficient implementation with output quality.

In order to perform this tradeoff, we treat the special case of a linear hardware cost metric (which can be applied to adders and multipliers). We model the effect of truncation as the injection of additive white uniform noise, which leads to a convenient performance metric. The algorithm wordlengths are determined by minimizing one of these two metrics subject to constraints on the other. In our previous work [5] we developed an ad hoc method for this optimization; in this paper, we derive closed-form expressions for the optimal wordlengths, which would be suitable for any FPGA or ASIC technology. Additionally, we then show that these expressions can be efficiently calculated in hardware, leading to wordlengths that change in real-time as a function of the data. We show that this result can be directly applied to several adaptive FIR filter architectures.

2. CLOSED-FORM SOLUTION

We consider two possible formulations. In the first, the hardware cost is minimized subject to constraints on the output noise variance. In the second, the output noise variance is minimized subject to a hardware constraint. In both cases, the unknown variable is the vector of wordlengths $\mathbf{b} = [b_1, b_2, \dots, b_N]^T$ to be assigned at the desired locations in the algorithm. The hardware cost function $C(\mathbf{b})$ and the output noise variance $V(\mathbf{b})$ takes the form

$$C(\mathbf{b}) = \sum_{i=1}^N c_i b_i, \quad V(\mathbf{b}) = \sum_{i=1}^N \alpha_i 2^{-2b_i}. \quad (1)$$

The form of $V(\mathbf{b})$ results from a linearization of the algorithm that is being mapped about an operating point. Essentially, the α_i are scaled versions of the maximum squared slope of the transfer function from the point of rounding to the algorithm output. The 2^{-2b_i} term results from assuming uniform additive noise [6]. Notice that $C(\mathbf{b})$ is a linear function of \mathbf{b} , and $V(\mathbf{b})$ is a convex function of \mathbf{b} . We will assume that the cost coefficients c_i , and the variance injection scale factors α_i are known.

2.1. Minimize Cost Subject to Variance Constraints

In this section we consider the problem of minimizing the hardware cost subject to a constraint on the maximum allowable noise variance:

$$\begin{aligned} &\min C(\mathbf{b}), \\ &\text{subject to } V(\mathbf{b}) \leq A \text{ and } \mathbf{b} \geq \mathbf{b}_{\min} \geq 0. \end{aligned} \quad (2)$$

Notice that we have not imposed integrality on the b_i 's, and that this is the optimization of a linear function over a convex set.

To obtain a closed-form expression, we first relax the minimum wordlength constraints in (2). Using a Lagrange multiplier, we minimize

$$J = \sum_{i=1}^N c_i b_i + \lambda \sum_{i=1}^N \alpha_i 2^{-2b_i}. \quad (3)$$

Taking the derivative and setting equal to zero yields

$$\frac{\partial J}{\partial b_k} = c_k + \lambda \alpha_k 2^{-2b_k} \ln(2)(-2) = 0, \quad (4)$$

which gives

$$\lambda = \frac{c_k}{\alpha_k 2 \ln(2)} 2^{2b_k}, \quad k = 1, \dots, N. \quad (5)$$

We can thus equate the right-hand side of (5) for different values of k , specifically

$$\frac{c_1}{\alpha_1} 2^{2b_1} = \frac{c_k}{\alpha_k} 2^{2b_k}. \quad (6)$$

Rearrangement gives

$$\left(\frac{c_k}{c_1}\right) \left(\frac{\alpha_1}{\alpha_k}\right) = 2^{2(b_1 - b_k)}. \quad (7)$$

Taking logarithms and rearranging finally gives

$$b_k = b_1 + \frac{1}{2} \log_2 \left(\frac{c_1}{\alpha_1} \frac{\alpha_k}{c_k} \right) \quad k = 1, \dots, N. \quad (8)$$

Thus, (8) allows us to find all the wordlengths from knowledge of b_1 . We can actually derive b_1 as well as the remaining b_k by substituting (8) into (2):

$$\begin{aligned} A &= \sum_{i=1}^N \alpha_i 2^{-2b_i} \\ &= \sum_{i=1}^N \alpha_i 2^{-2 \left(b_1 + \frac{1}{2} \log_2 \frac{c_1}{\alpha_1} \frac{\alpha_i}{c_i} \right)} \\ &= 2^{-2b_1} \frac{\alpha_1}{c_1} \sum_{i=1}^N c_i. \end{aligned} \quad (9)$$

Solving for b_1 ,

$$b_1 = \frac{1}{2} \log_2 \left(\frac{\alpha_1}{c_1} \frac{1}{A} \sum_{i=1}^N c_i \right). \quad (10)$$

From (8) we therefore have

$$b_k = \frac{1}{2} \log_2 \left(\frac{\alpha_1}{c_1} \frac{1}{A} \sum_{i=1}^N c_i \right) + \frac{1}{2} \log_2 \left(\frac{c_1}{\alpha_1} \frac{\alpha_k}{c_k} \right). \quad (11)$$

Finally, we simplify this expression as

$$b_k = \frac{1}{2} \log_2 \left(\frac{\alpha_k}{c_k} \right) + \frac{1}{2} \log_2 \left(\frac{\sum c_i}{A} \right). \quad (12)$$

It is important to note that (12) separates the wordlength formula into a variable portion (e.g. changes with k), and a portion that is common to all b_k .

A shortcoming of (12) is that the wordlengths may violate the minimum wordlength constraints of (2). We resolve this by an iterative approach in which those b_k 's that violate the constraint are fixed to be $b_{min,k}$, and then (12) is applied again to those variables which are not already assigned. Another problem is that the b_k 's are not necessarily integral. This is easily remedied by taking the ceiling of the expressions.

2.2. Minimize Variance Subject to Cost Constraints

Now we consider the problem of minimizing the roundoff noise variance subject to a constraint on the maximum allowed hardware cost:

$$\begin{aligned} &\min V(\mathbf{b}), \\ &\text{subject to } C(\mathbf{b}) = C \text{ and } \mathbf{b} \geq \mathbf{b}_{min} \geq 0. \end{aligned} \quad (13)$$

Notice that again we have not imposed integrality on the b_i 's, and that this is the optimization of a convex function over a linear space, for which well-defined optimality conditions exist.

Relaxing the minimum wordlength constraint of (13) and using a Lagrange multiplier, we obtain exactly the same

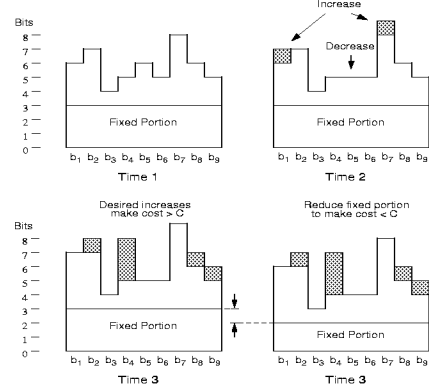


Figure 1. Adaptively varying wordlengths.

expression as in (8). We now perform manipulations similar to those used to derive (12), obtaining

$$b_k = \frac{1}{2} \log_2 \left(\frac{\alpha_k}{c_k} \right) + \frac{C + \frac{1}{2} \sum c_i \log_2 \frac{c_i}{\alpha_i}}{\sum c_i}. \quad (14)$$

We again ensure that the wordlength allocations do not violate the minimum wordlength constraints of (13), and also ensure integrality by taking the floor of the results.

3. REAL-TIME WORDLENGTH ADAPTATION

Comparing (14) to (12), we see that the components that vary with k are identical. We can take advantage of this for an adaptive computing implementation, where the wordlengths are to be calculated as a function of the data in real-time.

For example, suppose we are operating in a regime where the costs c_i , the variance injection parameters α_i and the desired variance A are such that reasonable wordlengths result. In this case, the total cost will be less than C . The design method will produce b_k 's that minimize the total hardware cost, allowing perhaps other hardware algorithms to share the hardware computing resources, or perhaps allowing for a lower power utilization. If only the α_k 's are allowed to vary, then the common term in (12) remains constant and can be precalculated.

If the data becomes worse in some way, causing the c_k and α_k 's to be less favorable, the total cost may exceed C . In this case we could use (14) to calculate the wordlengths. However, because the k -varying portions of (12) and (14) are identical, the common term in (14) must act to uniformly reduce the wordlengths so that the total cost is less than C . We thus do not need to calculate that common term; we merely reduce each wordlength by the same amount until the cost target is met. In this regime, we will not be able to meet the noise variance target A , but we will come as close as we can given the constraint on total available hardware.

Figure 1 illustrates this idea. At time 1 in the figure, certain wordlengths are allocated. At time 2, we increase some of the wordlengths according to (12). At time 3, we desire to further increase some wordlengths, but this would cause the total hardware cost to exceed the bound C . At this point, we uniformly decrease the common portion for all wordlengths until the total cost bound is met. If the data improves so that we are substantially below our cost boundary, then we uniformly add bits back to the common portion to improve our noise variance.

A simpler, approximate mechanization of this approach is to use two thresholds. We try to keep the filter wordlengths "between" the two thresholds. For example, we may require that maximum wordlength is between the thresholds

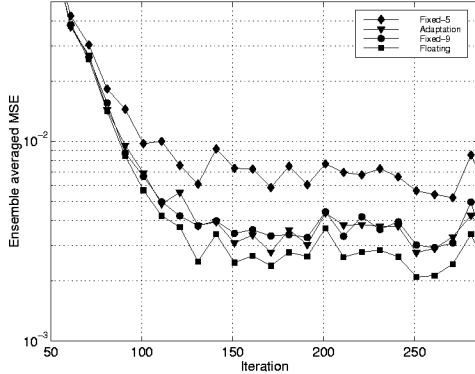


Figure 4. Comparison of LMS performance for four implementations.

mean-squared-error as a function of time for four implementations of the LMS algorithm: floating-point, $b_k = 9$ fixed-point, $b_k = 5$ fixed-point, and finally, real-time wordlength adaptation with the time constraint $\sum b_k \leq 55$.

From the figure, the floating-point implementation gives the lowest average MSE, and the 9-bit fixed-point implementation comes very close to it. Cutting back to 5 bits at the inputs gives rise to significantly higher MSE. However, by using wordlength adaptation, we were able to match the performance of the 9-bit implementation using only an average of 5 bits at the input.

Figure 5 shows the sum of the bit allocations at each LMS iteration for different constraint levels of 33, 55, or 77 total computing cycles per output. While the constraint is not met at every iteration, the average time per output sample does meet the constraints. More sophisticated methods of setting $q[n]$ in (20) can of course reduce the amount of variance in the total number of allocated bits in each cycle.

To further illustrate that the wordlength adaptation process does indeed allocate reasonable wordlengths, Figure 6a shows b_k averaged over time during the last 100 iterations for varying hardware constraints. In each case, the bit allocations were constrained to use at least 2 bits. Figure 6b shows the impulse response to which the adaptive filters converged. By comparing Figures 6a and 6b, we see that more bits are allocated to those inputs multiplying taps with large absolute values, just as we expected.

6. DISCUSSION AND FUTURE DIRECTIONS

In this paper we were concerned with optimizing both the wordlengths and performance of hardware-based algorithms. By restricting the cost function to the simple but important linear case, we were able to derive closed-form expressions for the wordlengths. Additionally, the form of the expressions admitted an efficient hardware implementation to allow the real-time calculation of the wordlengths. We applied this to a simple bit-serial implementation of an LMS adaptive filter.

Higher speed, parallel implementations of the LMS filter are also of great interest. We are currently investigating a fully parallel implementation, where truncating the input data results not in a hardware savings, but rather in a power savings, due to a reduction in the number of logic transitions.

The approach outlined here is more generally applicable than to just linear filtering. We are currently examining optimal wordlength allocation for more general cases of nonlinear algorithms and nonlinear cost functions.

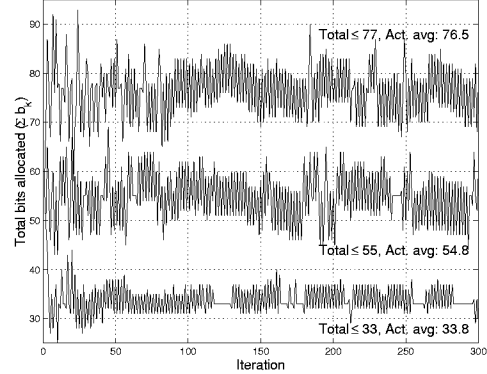


Figure 5. Total bits allocated using approximate wordlength adaptation procedure in each iteration for constraints of 33, 55, and 77 total bits.

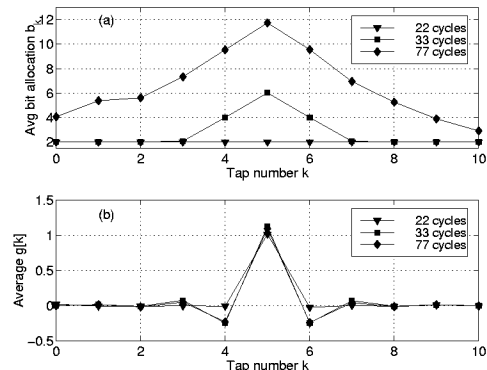


Figure 6. (a) Average number of bits allocated to each input position during the last 100 iterations of LMS processing. (b) Average tap weights during the last 100 iterations.

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