# HARMONIC RETRIEVAL IN COLORED NON-GAUSSIAN NOISE

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## ABSTRACT

This paper addresses the harmonic retrieval problem in colored linear non-Gaussian noise of unknown covariance and unknown distribution. The assumptions made in the reported studies, that the non-Gaussian noise is asymmetrically distributed and no quadratic phase coupling occurs ,are released. Using the elaborately defined fourth-order cumulants of the complex noisy observations which are obtained through Hilbert transform ,we can either estimate the noise correlation nonpapametrically via cumulant projections or obtain the AR polynomial of the non-Gaussian noise parametrically through ARMA modeling, then it is shown that the prewhitening or prefiltering techniques can be employed to retrieve harmonics respectively. Simulation results are presented to demonstrate the performance of the proposed algorithms.

## 1. INTRODUCTION

Colored non-Gaussian noise environments are frequently related to sonar systems and signal detection, see[3].More recently, harmonic retrieval in colored non-Gaussian noise with asymmetrical distribution was studied in [1][7].The key idea of these approaches is that the third-order cumulants of the non-Gaussian noise can be estimated in the presence of harmonics, consequently the correlation or the AR polynomial of the non-Gaussian noise is recovered ,then these estimated noise characteristics are used to whiten or partially whiten(make MA)the noise. Finally many conventional method can be applied to retrieve the harmonics.

It is well known that the third-order cumulants of quadratic phase coupled harmonics are nonzero[6]while the third-order cumulants of symmetrically distributed noise are zero. Although the fourth-order cumulants of symmetrically distributed noise are nonzero, those of the harmonics also do not vanish.As a result, noise characteristics can no longer be estimated in the presence of harmonics. The purpose of this paper is to resolve this problem. In section 3 we show that the complex counterpart of the real noisy observations can be obtained through Hilbert transform. Some particularly cumulant definitions were proved suitable for estimating noise characteristics. In section 4 two methods ,namely the prewhitening method and prefiltering method ,are proposed to retrieve harmonics. Simulation results are given to illustrate the performance of the new approaches in section 5.

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## 2.HARMONICS IN NON-GAUSSIAN NOISE: SIGNAL MODEL

We employ the discrete-time signal-plus-noise model given by

$$y(n) = s(n) + w(n) \quad (1)$$

where $s(n)$ is a real-valued harmonic signal given by

$$s(n) = \sum_{i=1}^{p} a_i \cos(\omega_i n + \varphi_i) \quad (2)$$

in (2) $a_i$'s and $\omega_i$'s are constants while $\varphi_i$'s are uniformly distributed over $[0,2\pi)$ . The problem of interest in this paper is to estimate $p$ and $\omega_i$'s using just the noisy observations $y(n)$ , $n = 1,\cdots,N$.

ASSUMPTION A1 The noise is given by $w(n) = \sum_\tau h_w(\tau)e(n-\tau)$ ,where $h_w(n)$ is a stable linear system with absolutely summable impulse response, $e(n)$ is a non-Gaussian independent, identically distributed driving sequence.

There are occasions to restrict noise model further to ARMA process.

ASSUMPTION A2 The noise $w(n)$ is modeled as an ARMA( $n_b, n_d$ ) process given by

$$B(q)w(n) = D(q)e(n) \quad (3)$$

where $B(q) = \sum_{j=0}^{n_b} b(j)q^{-j}, D(q) = \sum_{j=0}^{n_d} d(j)q^{-j}$ and $q^{-j}e(n) = e(n-j)$ .The ARMA process is assumed exponentially stable and free of pole-zero cancellations, is causal but may be nonminimum phase .

REMARK Contrasting with the reported studies, there is no assumption that no quadratic phase coupling occurs, and the additive non-Gaussian noise $w(n)$ may be either asymmetrically or symmetrically distributed.

## 3.ESTIMATION OF THE NON-GAUSSIAN NOISE CHARACTERISTICS

When the harmonics are quadratic phase coupled and/or the noise is symmetrically distributed, the noise characteristics can't be determined from both the third-order and fourth-order cumulants of the real noisy signal . Hence we transform the real observations into their complex counterpart. It is well known that for complex process there are many different cumulant definitions depending on the position of conjugations. Among them some particular definition, which is zero for coupled or uncoupled harmonics while is nonzero for symmetrically or asymmetrically distributed non-Gaussian noise, may be chosen to estimate the noise characteristics.

Given a real process $x(t)$ , its complex counterpart is

$$\tilde{x}(t) = x(t) + j\bar{x}(t) \quad (4)$$

where $\bar{x}(t)$ is the Hilbert transform of $x(t)$ . $\tilde{x}(t)$ can be seen as the output of a linear model with the input $x(t)$ , the impulse response of this linear model is

$$h_1(t) = \delta(t) - \frac{1}{j\pi t} \quad (5)$$

Obtain the complex counterpart of the real noisy signal $y(n)$ in (1) using Hilbert transform

$$\tilde{y}(n) = y(n) + j\bar{y}(n) = \tilde{s}(n) + \tilde{w}(n) \quad (6)$$

where $\tilde{s}(n)$ is the complex-valued harmonic signal

$$\tilde{s}(n) = \sum_{i=1}^{p} a_i \exp(j(\omega_i n + \varphi_i)) \quad (7)$$

$\tilde{w}(n)$ is the complex non-Gaussian noise. $\tilde{y}(n)$ can be depicted by the block diagram of Fig 1 .

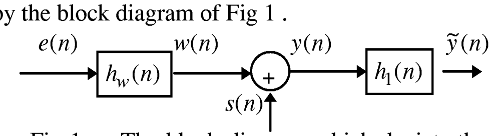


Fig 1. The block diagram which depicts the complex observations $\tilde{y}(n)$ .

Now we show the method of estimating the noise covariance under assumption A1 and the method of determining the noise AR parameters under assumption A2.

## 3.1 Estimating The Noise Covariance Using Cumulant Projections

Definition 1: The fourth-order cumulant of a complex process $\tilde{x}(n)$ is defined as

$$c_{4\tilde{x}}(\tau_1,\tau_2,\tau_3) = cum(\tilde{x}(n),\tilde{x}^*(n+\tau_1),\tilde{x}(n+\tau_2),\tilde{x}(n+\tau_3)) \quad (8)$$

It is easy to prove that according to definition 1,the fourth-order cumulant of quadratic phase coupled or uncoupled harmonics is zero, Thus

$$c_{4\tilde{y}}(\tau_1,\tau_2,\tau_3) = c_{4\tilde{w}}(\tau_1,\tau_2,\tau_3) \quad (9)$$

Under assumption A1, a simple relationship exists between the higher and lower order cumulants of $w(n)$ ,which is termed the projection property of cumulants [1],next we'll derive the complex counterpart of this relationship.

THEOREM 1: The cumulants of the complex non-Gaussian noise $\tilde{w}(n)$ in (6) according to definition 1 satisfy the following cumulants projection property

$$c_{2\tilde{w}}(\tau_1) = \alpha_{24} \sum_{\tau_2=-\infty}^{\infty} \sum_{\tau_3=-\infty}^{\infty} c_{4\tilde{w}}(\tau_1,\tau_2,\tau_3) \quad (10)$$

Where $c_{2\tilde{w}} = E(\tilde{w}(n)\tilde{w}^*(n+\tau_1))$ is the covariance of $\tilde{w}(n)$ , $c_{4\tilde{w}}$ according to definition 1 , $\alpha_{24}$ is a constant.

Proof : From Fig 1,the serial connection of $H_w(\omega)$ and $H_1(\omega)$ can be regarded as a linear system with transfer function $H(\omega) = H_w(\omega)H_1(\omega)$ ,its impulse response is

$$h(n) = h_w(n) * h_1(n) \quad (11)$$

then $\tilde{w}(n) = \sum_i e(i)h(n-i) \quad (12)$

from (5) and assumption A1, $h(n)$ is absolutely summable. It can be obtained that $c_{4\tilde{w}}$ according to definition 1 is

$$c_{4\tilde{w}}(\tau_1,\tau_2,\tau_3) = r_{4,e}\sum_{l=0}^{\infty} h(l)h^*(l+\tau_1)h(l+\tau_2)h(l+\tau_3)$$

then $\sum_{\tau_2=-\infty}^{\infty} \sum_{\tau_3=-\infty}^{\infty} c_{4\tilde{w}}(\tau_1,\tau_2,\tau_3)$

$$= r_{4,e}\sum_{l=0}^{\infty} h(l)h^*(l+\tau_1) \sum_{\tau_2=-\infty}^{\infty} h(l+\tau_2) \sum_{\tau_3=-\infty}^{\infty} h(l+\tau_3)$$

$$= \frac{r_{4,e}}{r_{2,e}} r_{2,e} \sum_{l=0}^{\infty} h(l)h^*(l+\tau_1)H^2(0) = \frac{r_{4,e}}{r_{2,e}} H^2(0)\ c_{2\tilde{w}}(\tau_1)$$

therefore, let $\alpha_{24} = \frac{\gamma_{2e}}{\gamma_{4e}}[H(0)]^{-2}$ ,it is easy to get (10).

It should also be noticed that the fourth-order cumulants according to some other definitions may not satisfy (10) such as

$$c_{4\tilde{x}}(\tau_1,\tau_2,\tau_3) = cum(\tilde{x}(n),\tilde{x}(n+\tau_1),\tilde{x}(n+\tau_2),\tilde{x}(n+\tau_3)) \quad (13)$$

From Theorem 1and (9) the noise covariance can be estimated in the presence of harmonics.

## 3.2 Determining The Noise AR Parameters

Under assumption A2,we'll show that the AR order $n_b$ and AR coefficients $b_i$ , $i = 1,\cdots,n_b$ can be determined in the presence of the quadratic phase coupled or uncoupled harmonics with no restriction on the non-Gaussian noise distribution.

Definition 2: The fourth-order cumulant of a complex process $\tilde{x}(n)$ , $c_{4\tilde{x}}(\tau_1,\tau_2,\tau_3)$ is defined as

$$c_{4\tilde{x}} = cum(\tilde{x}(n), \mathrm{Re}(\tilde{x}(n+\tau_1)), \tilde{x}(n+\tau_2), \tilde{x}(n+\tau_3)) \quad (14)$$

where $\mathrm{Re}(\tilde{x}(n+\tau_1)) = x(n+\tau_1)$ is the real part of $\tilde{x}(n+\tau_1)$ . It is obvious that according to definition 2

$$c_{4\tilde{x}}(\tau_1,\tau_2,\tau_3) = \frac{1}{2}(cum(\tilde{x}(n),\tilde{x}^*(n+\tau_1),\tilde{x}(n+\tau_2),\tilde{x}(n+\tau_3))$$

$$+ cum(\tilde{x}(n),\tilde{x}(n+\tau_1),\tilde{x}(n+\tau_2),\tilde{x}(n+\tau_3))) \quad (15)$$

Then we can easily obtain the result that according to definition 2 $c_{4\tilde{s}}$ is zero no matter whether the harmonics are quadratic phase coupled, whereas $c_{4\tilde{w}}$ is nonzero as long as $w(n)$ is non-Gaussian.

Theorem 2 : The fourth-order cumulants of the complex non-Gaussian noise $\tilde{w}(n)$ in (6) according to definition 2 satisfy the following higher order Yule-Walker equation

$$\sum_{i=0}^{n_b} b(i) C_{4,\tilde{w}}(\tau_1 - i,\tau_2,\tau_3) = 0 \qquad \tau_1 > n_d \quad (16)$$

Proof : The impulse response of the ARMA model in assumption A2 is $h_w(n)$ ,then

$$\mathrm{Re}(\tilde{w}(n)) = w(n) = \sum_{\tau} h_w(\tau) e(n-\tau) \quad (17)$$

According to definition 2 and from(12)(17),we have

$$C_{4,\tilde{w}}(\tau_1,\tau_2,\tau_3) = r_{4,e}\sum_{n=0}^{\infty} h(n)h_w(n+\tau_1)h(n+\tau_2)h(n+\tau_3) \quad (18)$$

then $\sum_{i=0}^{n_b} b(i)C_{4,\tilde{w}}(\tau_1 - i,\tau_2,\tau_3)$

$$= r_{4,e}\sum_{n=0}^{\infty} h(n)\left(\sum_{i=0}^{n_b} b(i)h_w(n+\tau_1 - i)\right)\ h(n+\tau_2)h(n+\tau_3)$$

notice $\sum_{i=0}^{n_b} b(i)h_w(n+\tau_1 - i) = d(n+\tau_1)$ ,where $d(i) = 0$ if $i > n_d$ or $i < 0$ .Consequently when $\tau_1 > n_d$ ,the left-hand side of (16) is identical with zero.

From theorem 2, the AR order and AR parameters can be determined by Giannakis and Mendel[4].It is well known that the fourth-order cumulant of a complex process has $2^4$ different definitions depending on the position of conjugations. The result can be obtained that none of these cumulant definitions alone satisfies the Yule-Walker equation in (16),thus they all can't be used to estimate the noise AR order and parameters.

# 4. HARMONIC RETRIEVAL

we propose two approaches for harmonic retrieval. One is the prewhitening method using noise covariance, the other is the prefiltering method using noise AR parameters.

## 4.1 Prewhitening Method

The covariance matrix of $\tilde{y}(n)$ in (6) is given by,

$$\mathbf{R}_{\tilde{y}} = E(\tilde{\mathbf{y}}(\mathbf{n})\tilde{\mathbf{y}}(\mathbf{n})^{\mathbf{H}}) = \mathbf{APA^H} + \sigma_{\tilde{w}}^2 \Sigma_{\tilde{w}} \quad (19)$$

$\sigma_{\tilde{w}}^2$ is the noise variance, $\Sigma_{\tilde{w}}$ is the normalized noise correlation matrix. If $\Sigma_{\tilde{w}} = \mathbf{I}$ then the noise is white and the problem reduces to conventional eigenanalysis. In the colored noise case, we may apply the result of [2] directly

$$\mathbf{R}_{\tilde{y}}\ \mathbf{e}_i = \lambda_i\ \Sigma_{\tilde{w}}\ \mathbf{e}_i \quad (20)$$

which is a generalized eigenvalue problem which may be solved using SVD. The resulting eigenvector estimates can then be used in the various noise subspace methods for harmonic retrieval. The difference between the new prewhitening method and that proposed in [1]lies in the estimation technique of the noise correlation matrix $\Sigma_{\tilde{w}}$ . Our new prewhitening method is summarized in the following steps.

1) Obtain $\tilde{y}(n)$ from $y(n)$ using Hilbert transform as in (6).

2) Estimate the covariance matrix $\mathbf{R}_{\tilde{y}}$ in (19).

3) Estimate $c_{4\tilde{y}}$ according to definition 1.

4) Estimate $\Sigma_{\tilde{w}}$ using cumulant projections via theorem 2 .

5) Solve (20) using the estimates from steps 2 and 4.

6) Use the resulting noise subspace eigenvectors in the conventional noise subspace method to retrieve the harmonics.

## 4.2 Prefiltering Method

Having determined the noise AR parameters $b(i)\ \ i = 1,\cdots,n_b$ , denote $\breve{y}(n) = B(q)y(n) = \sum_{i=0}^{n_b} b(i)y(n-i)$ .We refer to $\breve{y}(n)$ as the filtered output process. From [7],the correlation of $\breve{y}(n)$ satisfies the following modified Yule-walker (MYW) equation

$$\sum_{i=0}^{2p} a(i)r_{\breve{y}}(l-i) = 0 \quad l > 2p + n_d \quad (21)$$

and the polynomial $A(z) = \sum_{m=0}^{2p} a(m)z^{-m}$ has complex conjugate roots at $z = e^{\pm jw_i}\ (i = 1,2,\cdots,p)$ . The order $2p$ and parameters $a(i), i = 1,\cdots,2p$ can be estimated by using SVD-TLS method[5]

The following is our new prefiltering approach

1) Obtain $\tilde{y}(n)$ from $y(n)$ using Hilbert transform .

2) Estimate $c_{4\tilde{y}}$ according to definition 2,then determine $n_b$ and $b(i), i = 1,\cdots,n_b$ from (16) via SVD-TLS method.

3) Use the estimated AR polynomial to obtain $\breve{y}(n)$ .

4) Compute the correlation of $\breve{y}(n)$ ,then determine the coefficients $a(i), i = 1,\cdots,2p$ from (21).

5) Find roots of $A(z) = 0$ then compute the frequencies .

# 5. SIMULATION RESULTS

Simulation results are focused on demonstrating the effectiveness of the proposed methods, there is no comparison between them. In example 1, 2,the prewhitening method is employed while in example 3,4 the prefiltering method is used. Consider a time series of the form

$$y(n) = \sin(2\pi f_1 n + \varphi_1) + \sin(2\pi f_2 n + \varphi_2) + w(n) \quad (22)$$

The noise $w(n)$ in example1 and 2 is an AR(2) noise with AR parameters [1,0,0.9025] whose spectrum shows a sharp peak.

Example 1: $f_1 = 0.18$ , $f_2 = 0.36$ , $\varphi_1 = \pi/6, \varphi_2 = \pi/3$ ,the harmonics are quadratic phase coupled. SNR=-2dB, $w(n)$ is exponential (asymmetrically) distributed, the data length is 512.The method proposed in [1]estimate the noise correlation via third-order cumulant projections, its result is shown in Fig 2.a, while the result of the new method is shown in Fig 2.b.

Example 2: $f_1 = 0.2, f_2 = 0.35$ ,SNR=-2dB,the data length is 512 , $w(n)$ is mixed Gaussian(symmetrically) distributed. The results of the method in [1] and the new prewhitening method are shown in Fig 3.a and Fig 3.b respectively.

The method proposed in [1] can also estimate the noise correlation via fourth-order cumulant projections, but the result is biased. When the SNR is not appropriate or the kurtosis of the driving sequence of $w(n)$ is comparatively small, the bias will be large. In the next two examples ,the performance of the new prefiltering method is shown. The noise $w(n)$ is an ARMA process with AR parameters [1 -1.5 0.8] and MA parameters [1 -0.75 -2.5].

Example 3: $f_1 = 0.23,\ f_2 = 2f_1, \varphi_1 = \pi/6, \varphi_2 = \pi/3$, quadratic phase couple occurs , $w(n)$ is exponential distributed,SNR=-3dB,parameter estimation results of the new prefiltering method are summarize in Table I .

Example 4: $f_1 = 0.26, f_2 = 0.3$, $w(n)$ is mixed Gaussian distributed,SNR=-5dB,the results are given in Table II.

In last two examples ,the noise AR parameter estimates are heavily biased if using the prefiltering method proposed in[7],which results the frequency estimates are often unobtainable because there is no solution of $A(z) = 0$ .While the new prefiltering method can obtain satisfied results.

## 6.CONCLUSION

In this paper, two methods are proposed to retrieve harmonics in colored linear non-Gaussian noise especially when the noise is symmetrically distributed or the harmonics are quadratic phase coupled. In order to estimate the non-Gaussian noise characteristics in the presence of harmonics, Hilbert transform is used to transform the real observations into their complex form. Then the cumulants projection property and higher order Yule-Walker equation of this complex process are established, which are used in the proposed prewhitening and prefiltering method respectively. Simulation results have shown the effectiveness of the new methods.

## REFERENCE


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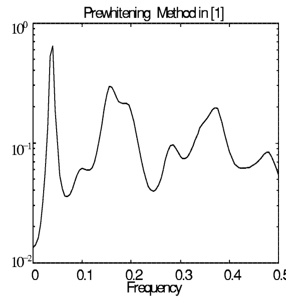


Fig 2.a

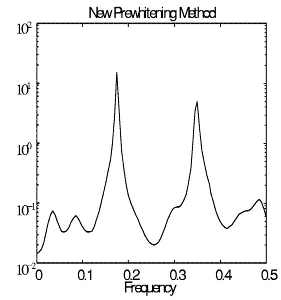


Fig 2.b

Fig 2.a is the result of the method in [1] in the case of quadratic phase coupled harmonics, while Fig 2.b is the result of the new prewhitening method.

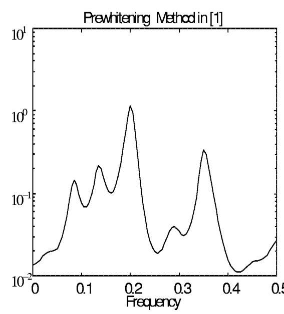


Fig 3.a

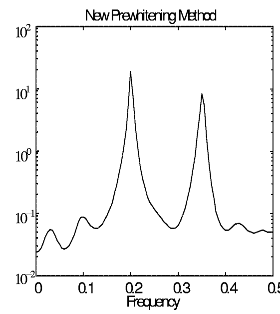


Fig 3.b

Fig 3.a is the result of the method in [1] in the case of symmetrically-distributed non-Gaussian noise, while Fig 3.b is the result of the new prewhitening method.

Table I

| Noise Model | b(1) | b(2) |
|---|---|---|
| True | -1.5 | 0.8 |
| Estimate | -1.5617(0.1457) | 0.8722(0.0693) |
| Frequency | $f_1$ | $f_2$ |
| True | 0.23 | 0.46 |
| Estimate | 0.2306(0.0008) | 0.4603(0.0009) |

Table II

| Noise Model | b(1) | b(2) |
|---|---|---|
| True | -1.5 | 0.8 |
| Estimate | -1.4773(0.1316) | 0.8338(0.0929) |
| Frequency | $f_1$ | $f_2$ |
| True | 0.26 | 0.3 |
| Estimate | 0.2601(0.0005) | 0.3002(0.0006) |

Statistics of AR parameter and frequency estimates obtained via the new prefiltering approach(N=2048 in each run, 20 Monte-Carlo Runs).Table I is for example 3 and Table II is for example 4.