

# ADAPTIVE WINDOW IN THE PWVD FOR THE IF ESTIMATION OF FM SIGNALS IN ADDITIVE GAUSSIAN NOISE

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## ABSTRACT

The peak of the polynomial Wigner-Ville distribution is known to be a consistent estimator of the instantaneous frequency for polynomial FM signals. In this paper, we present an algorithm for the design of an optimal time-varying window length for this estimator when noisy non-linear, not necessarily polynomial, FM signals are considered. The results obtained show that the estimator is accurate and outperforms any fixed window time-frequency distribution based estimator.

## 1. INTRODUCTION

For non-stationary signals, i.e., signals whose spectral contents vary with time, the frequency at a particular time is well described by the concept of instantaneous frequency (IF) [3]. In many real-life applications such as radar, sonar, bio-medical engineering, and automotive signals, the IF characterises important physical parameters of the signals [9]; therefore, it is desirable to have effective methods for IF estimation.

Two major approaches exist in the literature for IF estimation. One approach assumes a certain form of the signal and uses a mathematical model to estimate it. This is referred to as a parametric approach. Some problems limit its application in that it is difficult to find the “correct” mathematical model of the signal. Furthermore, signal parameters estimation becomes cumbersome as the order of non-linearity of the signal increases. Alternatively, one may use a non-parametric approach for IF estimation. That is, no mathematical model of the signal is assumed. A well known class of non-parametric methods is based on time-frequency analysis.

Time-frequency analysis was introduced as a tool to characterise the time-varying spectral contents of non stationary signals. It is capable of displaying the temporal localisation of the signal’s spectral components, i.e., it is very powerful in IF localisation and estimation.

The Wigner-Ville distribution (WVD), a member of a family of bilinear time-frequency distributions [5], was shown to be efficient in the estimation of a linearly frequency modulated (FM) signal [8]. However, this property is no longer valid for non-linear FM signals. For this

type of signals various higher-order time-frequency distributions have been introduced. One of them is the polynomial Wigner-Ville distribution [4].

The polynomial Wigner-Ville distribution (PWVD) gives maximum energy concentration along the IF law for polynomial FM signals. For these signals, it was shown that the IF estimator using the peak of the PWVD is unbiased [2]. For non-polynomial FM signals, this estimator is biased. The bias is caused by three effects: the implementation error, the error due to noise, and the error due to the mismatch of the signal with the distribution order.

In this paper, and taking into consideration the above mentioned effects, we derive an expression for the bias as well as the asymptotic variance of the IF estimator for a non-linear, not necessarily polynomial, FM signal embedded in white Gaussian noise. It is shown that these expressions are highly signal dependent and tend to vary inversely in function of the window length of the time-frequency distribution. Based on this observation, we also derive an expression for the optimal window length that minimises the mean square error for the IF estimator and propose an algorithm to design the “best” PWVD in the sense of resolving the bias-variance trade-off. This work is an extension of the works in [6, 7] where the windowing in the WVD and the spectrogram were considered.

Simulation results for different highly non-linear FM signals show that the proposed algorithm can estimate the signal IF accurately.

The paper is organised as follows. In Section 2, we derive the expression for the optimal window. In Section 3, we present the algorithm for the choice of the best window for the PWVD. Some examples are presented in Section 4; whereas, Section 5 concludes the paper.

## 2. OPTIMAL WINDOW LENGTH DERIVATION

In this section we give the expressions of the bias and the variance of the IF estimator of a non-linear FM signal.

Consider the problem of IF estimation from the discrete-time observations

$$y(nT) = z(nT) + \epsilon(nT)$$

where  $n$  is an integer and  $T$  is a sampling period.  $z(nT)$  is a discrete-time version of the signal  $z(t) = A \cdot \exp(j\phi(t))$ ,

whereas,  $\epsilon(nT)$  is discrete-time complex-valued white Gaussian noise with i.i.d real and imaginary parts and whose variance is equal to  $\sigma_\epsilon^2$ .

The discrete-time domain definition of the windowed PWVD is given by

$$W_z^{(q)}(t, \omega) = \sum_{n=-\infty}^{\infty} w_h(nT) \prod_{i=1}^{q/2} z(t+c_i nT) z^*(t-c_i nT) e^{-j\omega nT} \quad (1)$$

where  $q$  is an even integer which indicates the order of non-linearity of the PWVD, and the coefficients  $c_i$  ( $i = 1, 2, \dots, q/2$ ) are calculated so that the PWVD is optimal for representing polynomial FM signals in the sense that it yields delta functions around the signal's IF. In (1),  $w_h(nT) = T/h \cdot w(nT/h)$  where  $w(t)$  is a real-valued symmetric window with  $w(t) = 0$  for  $|t| > 1/2$ ,  $h > 0$ .

Note that the WVD, which is optimal for linear FM signals only, is a member of the class of the PWVDs with parameters  $q = 2$  and  $c_1 = 0.5$ .

Replacing in (1) the signal  $z(t)$  by its expression and using Taylor's expansion of the phase,  $\phi(t)$ , around  $t$ , we obtain

$$W_z^{(q)}(t, \omega) = A^q \sum_{n=-\infty}^{\infty} w_h(nT) e^{jnT\phi'(t)} e^{j\Delta\phi(t, nT)} e^{-j\omega nT} \quad (2)$$

where  $\phi'(t)$  is the signal's IF and  $\Delta\phi(t, nT)$  is given by

$$\begin{aligned} \Delta\phi(t, nT) &= 2 \sum_{i=1}^{q/2} \sum_{s=1}^{\infty} \frac{(c_i nT)^{2s+1}}{(2s+1)!} \phi^{(2s+1)}(t) \\ &= 2 \sum_{s=1}^{\infty} \frac{(nT)^{2s+1}}{(2s+1)!} \phi^{(2s+1)}(t) \cdot \sum_{i=1}^{q/2} c_i^{2s+1} \end{aligned}$$

Note that for a given polynomial phase signal of order  $p$ , the PWVD is designed such that  $\sum_{i=1}^{q/2} c_i^m = 0$  for odd values of  $m$ , ( $3 \leq m \leq p$ ). Therefore, for this type of signals, and using the appropriate PWVD, we see that  $\Delta\phi(t, nT)$  is always zero through either  $\sum_{i=1}^{q/2} c_i^{2s+1} = 0$  (for  $3 \leq 2s+1 \leq p$ ) or  $\phi^{(2s+1)}(t) = 0$  (for  $2s+1 > p$ ). That is, the PWVD yields delta functions around the signal's IF. This suggests the use of the peak of the PWVD as an IF estimator. However, if the signal is not a polynomial FM signal, or if there is a mismatch between the polynomial FM signal and the PWVD order that we use, then an error is always present in the IF estimate. In addition to this, and in the case of a noisy signal, there is also a statistical error. In what follows, we will compute the expressions of the bias and the variance of such an error.

The IF estimate of the signal is given by the frequency where the maximum of the PWVD occurs, i.e.,

$$\hat{\omega} = \arg[\max_I W_y^{(q)}(t, \omega)]$$

with  $I = \{\omega : 0 \leq |\omega| \leq \pi/T\}$ . That is, the IF estimate is given by solving for  $\omega$

$$\frac{\partial W_y^{(q)}(t, \omega)}{\partial \omega} = 0$$

The linearisation of the above expression with respect to the effects mentioned above gives

$$\begin{aligned} \frac{\partial W_y^{(q)}(t, \omega)}{\partial \omega} &= \frac{\partial W_y^{(q)}(t, \omega)}{\partial \omega} \Big|_0 + \frac{\partial^2 W_y^{(q)}(t, \omega)}{\partial \omega^2} \Big|_0 \Delta\omega \\ &\quad + \frac{\partial W_y^{(q)}(t, \omega)}{\partial \omega} \Big|_0 \delta\Delta\phi + \frac{\partial W_y^{(q)}(t, \omega)}{\partial \omega} \Big|_0 \delta\epsilon \end{aligned}$$

where  $|_0$  means that the expressions are computed at the point where  $\omega = \phi'(t)$ ,  $\Delta\phi(t, nT) = 0$ , and  $\epsilon(nT) = 0$ , and  $\Delta\omega$  is the error in the estimate.

After evaluation of the above expressions, we find the bias and the variance to be

$$\text{Bias}(\hat{\omega}) = \frac{1}{h^2 F} \sum_{s=1}^{\infty} \phi^{(2s+1)}(t) B_h(s) \sum_{i=1}^{q/2} c_i^{2s+1} \quad (3)$$

$$\text{Var}(\hat{\omega}) = \frac{\sigma_\epsilon^2 \cdot \sum_{i=1}^M k_i^2}{2A^2} \cdot \frac{TE}{h^3 F^2} \quad (4)$$

with

$$F = \int_{-1/2}^{1/2} w(t) \cdot t^2 dt \quad E = \int_{-1/2}^{1/2} w^2(t) \cdot t^2 dt$$

and

$$B_s(s) = \frac{2h^{2s+2}}{(2s+1)!} \int_{-1/2}^{1/2} w(t) \cdot t^{2s+2} dt$$

where  $M$  is the number of coefficients in the kernel of the PWVD given in (1) and  $k_i$  is the multiplicity of each of these coefficients.

We can also note here, that for a given polynomial phase signal of order  $p$ , and using the appropriate PWVD, the bias in (3) is always zero through either  $\sum_{i=1}^{q/2} c_i^{2s+1} = 0$  (for  $3 \leq 2s+1 \leq p$ ) or  $\phi^{(2s+1)}(t) = 0$  (for  $2s+1 > p$ ).

If we choose the window to be rectangular, simple calculation shows that the mean square error of the IF estimate is found to be

$$\text{m.s.e}(\hat{\omega}) = \frac{6\sigma_\epsilon^2 \cdot \sum_{i=1}^M k_i^2 \cdot T}{A^2 h^3} + \left( \frac{\phi^{(3)}(t)}{20} h^2 \sum_{i=1}^{q/2} c_i^3 \right)^2 \quad (5)$$

In the above expression, we approximated the bias in (3) by its first term (i.e.  $s = 1$ ) which is non-zero only for the WVD. However, for a higher-order PWVD this term is zero (as well as all the terms up to the appropriate order of the PWVD used) and we should then approximate the bias by the first non-zero term in the sum (as explained in the previous paragraph). For example, the first non-zero term in the PWVD of order six [2] corresponds to  $s = 2$  and for the PWVD of order eight ([1]) it corresponds to  $s = 3$  and so on. Furthermore, this correction in the bias approximation would change only the expression of the mean square error in (5) (and the expression of the optimal length that follows) and will not affect the rest of the paper.

From equation (5), the optimal window length, for minimum m.s.e, for IF estimation is

$$h_{\text{opt}}(t) = \left[ \frac{1800\sigma_\epsilon^2 \cdot \sum_{i=1}^M k_i^2 \cdot T}{A \sum_{i=1}^{q/2} c_i^3 \phi^{(3)}(t)} \right]^{1/7}$$

As can be seen from above, the window length should be larger for signals with high phase variations. We must also note that if the IF of the signal is unknown, which is the most general case, then the expression for the optimal window length above is of no practical importance. In that case, and in order to avoid this difficulty, we propose in the next section an algorithm that can determine the adaptive optimal window length without having to know the IF and the signal to noise ratio.

### 3. ALGORITHM FOR THE ADAPTIVE WINDOW LENGTH

In this section, we propose a signal dependent time-varying window length that uses only the asymptotic variance expression and does not need any information about the signal's IF.

Due to space limitation, we will briefly state the algorithm here. However, the basic idea and its details can be found in [6, 7].

Asymptotically, the IF estimate  $\hat{\omega}$  is a random variable distributed around the true IF  $\phi'(t)$  with a bias,  $Bias(t, h)$ , and a standard deviation,  $\sigma_h$ . Thus, we may write

$$|\phi'(t) - \hat{\omega} + Bias(t, h)| \leq \kappa \sigma_h$$

where the inequality holds with a probability  $P(\kappa)$  depending on the parameter  $\kappa$ .

Consider a set of discrete window length values,  $h \in H$ ,

$$H = \{h_s | h_s = 2h_{s-a}, s = 1, 2, \dots, J\}$$

where  $h_0$  being a multiple of 2. Let us define the confidence intervals of the IF estimate as  $D_s = [L_s, U_s]$  with

$$L_s = \hat{\omega}_{h_s}(t) - (\kappa + \Delta\kappa)\sigma(h_s) \quad (6)$$

$$U_s = \hat{\omega}_{h_s}(t) + (\kappa + \Delta\kappa)\sigma(h_s) \quad (7)$$

where  $\hat{\omega}_{h_s}(t)$  is an estimate of the IF for the window length  $h = h_s$  and  $\sigma(h_s)$  its standard deviation. Let the window length  $h_{s+}$  correspond to the largest  $s$  ( $s = 1, 2, \dots, J-1$ ) when two successive confidence intervals still intersect, i.e., when  $D_s \cap D_{s+1} \neq \emptyset$  is still satisfied. Therefore, there exist values of  $\kappa$  and  $\Delta\kappa$  such that  $D_s \cap D_{s+1} \neq \emptyset$  and  $D_{s+1} \cap D_{s+2} = \emptyset$  for  $s = s^+$  when  $h_{s+} = h_{\text{opt}}$ . That is, the optimal window length is defined as the window for which two successive intervals no longer intersect.

The proof of the above proposition and the calculation of the values of  $\kappa$  and  $\Delta\kappa$  is well detailed in [6, 10].

Note that the search for the optimal window length over the finite set  $H$  is a simple optimisation problem. However, the discretisation of the window length  $h$  inevitably leads to a suboptimal window length value. Fortunately, this loss of accuracy is not significant in many cases as the mean square error has a stationary point for the optimal window length and varies very slowly for window lengths close to it.

The algorithm can now be stated as follows:

### 4. EXAMPLES AND RESULTS

For space reasons, only two examples are considered in this section. For the first one, the IF is defined as  $\omega(t) =$

For every time instant  $t$ ,

1. Compute a slice of the PWVD for every value of  $h_s \in H$ . Thus, we obtain

$$\{W_{h_s}^{(q)}(t, \omega)\} \quad \text{for every window } h_s \in H$$

The IF estimates (corresponding to every window length) are found as the maximum of each of these slices, i.e.,

$$\hat{\omega}_{h_s}(t) = \arg[\max_j W_{h_s}^{(q)}(t, \omega)]$$

2. The confidence intervals are computed for each window  $h_s$  using (6-7) with the standard deviation given by (4) and the amplitude as well as the noise variance estimated from the signal data as in [6].

3. The optimal window length is obtained as the first window in the set  $H$  when the inequality

$$|\hat{\omega}_{h_s}(t) - \hat{\omega}_{h_{s+1}}(t)| \leq (\kappa + \Delta\kappa)[\sigma(h_s) + \sigma(h_{s+1})]$$

is not valid anymore and the adaptive IF estimate is the IF that corresponds to this particular value of the window length.

4. Repeat the above steps for every time instant  $t$ .

40  $\text{asinh}(100t) + 256\pi$  and for the second one it is given by  $\omega(t) = 256\pi + 128\pi \text{sign}(\sin(2\pi t)) \cdot |\sin(2\pi t)|^{1/4}$ .

For both examples, the peak of the sixth order PWVD, defined by (1) with  $q = 6$  and

$$\prod_{i=1}^{q/2} z(t + c_i nT) z^*(t - c_i nT) = z(t + 0.62nT) z^*(t - 0.62nT) \\ z(t + 0.75nT) z^*(t - 0.75nT) \\ z(t - 0.87nT) z^*(t + 0.87nT),$$

is chosen as an IF estimator. The signal to noise ratio is chosen equal to 10 dB and the window set considered is  $H = \{2, 4, 8, \dots, 256\}$ .

The algorithm estimates the amplitude as well as the noise variance from the noisy signal. For this reason, an interpolation is necessary to obtain good estimates of these quantities.

In Fig.1, we plot the IF estimates for the first signal using respectively a small, large, and the adaptive window length which was found using the above algorithm. For the small window length, the variance is large in agreement with (4). However, for large variations in the phase, the bias is large in agreement with (3). Note the superiority of the adaptive window length in reducing the variance and the bias. The dashed lines represent the true IF.

In Fig.2, we plot the results for the second signal where the same conclusions can be inferred.

## 5. CONCLUSION

In this paper, we proposed a nonparametric method for the IF estimation of a non-linear FM signal. This method is based on the design of an adaptive optimal window length that minimises the mean square error of the estimator when the peak of the PWVD is used. Examples, of highly non-linear FM signals, show that the estimator is very accurate and outperforms any other fixed window time-frequency distribution.

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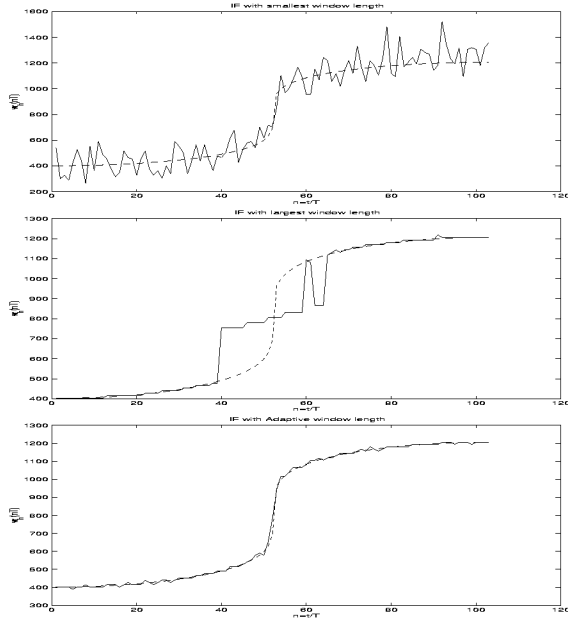


Figure 1: The IF estimates of the first signal for a small, large, and the adaptive window length respectively.

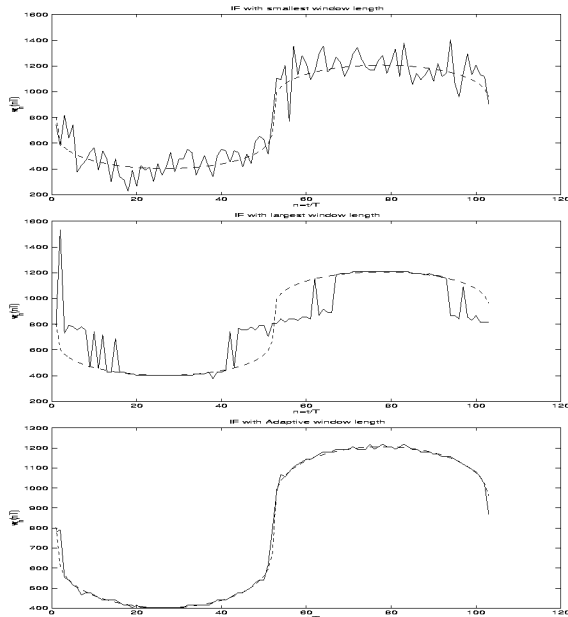


Figure 2: The IF estimates of the second signal for a small, large, and the adaptive window length respectively.