AN OPTIMAL SET-THEORETIC BLIND DECONVOLUTION SCHEME BASED ON HYBRID STEEPEST DESCENT METHOD

Isao Yamada, Masanori Kato and Kohichi Sakaniwa

Dept. of Electrical & Electronic Eng., Tokyo Institute of Technology,
2-12-1, Ookayama, Meguro-ku, Tokyo, 152-8552, Japan,
E-mails: isao@ss.titech.ac.jp, masa@ss.titech.ac.jp and sakaniwa@ss.titech.ac.jp

ABSTRACT

In this paper, we propose a simple set theoretic blind deconvolution scheme based on a recently developed convex projection technique called *Hybrid Steepest Descent Methods*. The scheme is essentially motivated by Kundur and Hatzinakos's idea that minimizes a certain cost function uniformly reflecting all *a priori* information such that (i) nonnegativity of the true image and (ii) support size of the original object.

The most remarkable feature of the proposed scheme is that the proposed one can utilize each *a priori* information separately from other ones, where some partial informations are treated in a set theoretic sense while the others are incorporated in a cost function to be minimized.

1. INTRODUCTION

In many image processing applications, the degradation of an image can be represented as the convolution of the true image with a blurring function known as a point-spread-function (PSF). The blurred image can be modeled as

$$g(x, y) = h(x, y) * f(x, y) + n(x, y), \tag{1}$$

where (x, y) : discrete pixel coordinates of the image frame, $g(x, y)$: blurred image, $f(x, y)$: true image, $h(x, y)$: point spread function (PSF), $n(x, y)$: additive noise, $*$: discrete two-dimensional (2-D) linear convolution operator.

Although the effect of PSF is usually assumed to be explicitly known in classical image restoration techniques to recover the true image $f(x, y)$, it is well known that accurate measurement of the degradation is often difficult, costly, dangerous, or physically impossible for example in applications such as astronomical speckle imaging and certain medical imaging etc. This situation motivated a notion of *Blind image restoration* that estimate both the true image and PSF simultaneously.

As seen in broad reviews on the blind deconvolution problem [1, 2], numerous strategies have been proposed to tackle this problem because of its great importance in applications as well as in theoretical interest.

In particular, recently, Kundur and Hatzinakos reported that the blind image deconvolution problem is successfully resolved by constructing a restoration filter with a little *a priori* information such that (i) nonnegativity of the true image and (ii) support size of the original object, where the true image is estimated by minimizing a certain cost function uniformly reflecting the all *a priori* information [3]. However, since each *a priori* information is a interpretation of absolutely required different physical constraints, separate and flexible use after examining the role of each information would be more desirable to the problem. For example, only nonnegativity of the filtered image is required to be satisfied over the support while the complete *a priori* information on the signal value is known as a background grey-level outside the support. This implies that the set theoretic strategy [4] is natural to utilize the *a priori* information over the support while optimization is suitable to do outside the support.

In this paper, motivated by the idea shown by Kundur and Hatzinakos, we propose a simple blind deconvolution scheme based on a recently developed convex projection technique called *Hybrid Steepest Descent Methods* [5, 6]. The most remarkable feature of the proposed scheme is that the proposed one can utilize each *a priori* information separately from other ones, where some partial informations are treated in a set theoretic sense while the others are incorporated in a cost function to be minimized.

In addition, some variants of the proposed method can still be applied to the blind deconvolution problems in which an inconsistent set of *a priori* informations is imposed. In such a case, these methods lead to the unique optimal FIR restoration filter among all FIR filters that attain the least sum of squared distances to all sets defined by each information.

A couple of simple numerical examples are presented to demonstrate the performance of the proposed blind deconvolution scheme in noisy case as well as in noiseless case.

2. REVIEW OF A NONPARAMETRIC BLIND DECONVOLUTION SCHEME

In this section, we present a brief review of the idea of a nonparametric blind deconvolution scheme proposed by Kundur and Hatzinakos [3].

Assume that the following *a priori* information on the imaging process, the true image, and the PSF.

1. The degradation of the true image is modeled by (1)
2. The object is imaged such that it is entirely encompassed by the observed frame.
3. The true image is nonnegative, and its support is known *a priori*, the support is defined to be the smallest rectangle encompassing the object.

4. The background of the image is uniformly grey, black, or white.
5. Fourier Transformation of PSF $H(\omega_1, \omega_2)$ satisfy the following condition $H(0,0) = 1$.
6. The inverse of the PSF exists, and both the PSF and its inverse are absolutely summable.

Most of these are commonly assumed in numerous deterministic blind deconvolution problems. The validity and broad availability of these assumptions are briefly discussed in [3].

The essential strategy of Kundur and Hatzinakos's scheme is best approximating the role of FIR restoration filter $\{u(x,y)\}$ to that of the inverse of the PSF over the support by applying part of the above *a priori* information. In other words, the consistency of the obtained estimate $\hat{f}(x,y) := u(x,y) * g(x,y)$ of the true image with these *a priori* informations is adaptively updated by minimizing a convex cost function:

$$J(u) = \sum_{(x,y)\in D_{\sup}} \hat{f}^2(x,y)\,\mathrm{cl}(\hat{f}(x,y)) + \sum_{(x,y)\in \overline{D_{\sup}}} [\hat{f}(x,y) - L_B]^2 + \gamma \left[\sum_{\forall(x,y)} u(x,y) - 1 \right]^2 \quad (2)$$

where

$$\mathrm{cl}(f) := \begin{cases} 0, & \text{if } f \geq 0 \\ 1, & \text{if } f < 0, \end{cases}$$

$D_{\sup}$ is the set of all pixels inside the region of support, $\overline{D_{\sup}}$ is the set of all pixels outside the region of support, L_B is the background grey-level value (if the background is black, then $L_B = 0$), and γ is introduced as a sort of penalty. Obviously the first term in (2) measures the consistency with the above 3rd assumption, the second term measures one with the 4th assumption, the third term reflect the 5th and 6th assumptions. The third term is additionally introduced in [3] to constraint the parameter $\{u(x,y)\}$ from the trivial all-zero solution when the background is black (i.e. $L_B = 0$).

Fig. 1 illustrates the principal strategy of the blind deconvolution scheme proposed in [3], where

$$\hat{f}_{NL}(x,y) := \begin{cases} \hat{f}(x,y), & \text{if } \hat{f}(x,y) \geq 0,\ (x,y) \in D_{\sup} \\ 0, & \text{if } \hat{f}(x,y) < 0,\ (x,y) \in D_{\sup} \\ L_B, & \text{if } (x,y) \in \overline{D_{\sup}} \end{cases}$$

It should be noted that, in the problem, only nonnegativity of the filtered image is absolutely required to be satisfied over the support while the complete *a priori* information is known as a background grey-level outside the support. However, the nonnegativity $\hat{f}(x,y)$ over the support is not guaranteed in general by minimizing J of (2). In addition, the final compensation by operating function f_{NL} to the part of image $\hat{f}(x,y)$ over the support seems debatable because it conflict with the principal strategy of best approximating the role of FIR filter over the support to that

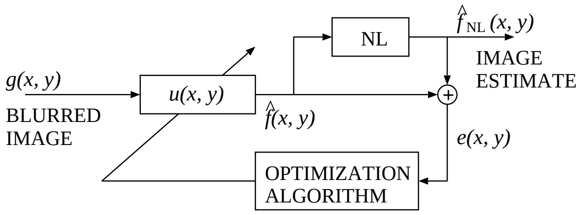


Figure 1: Kundur and Hatzinakos's Blind image deconvolution

of the inverse of the PSF. On the other hand, the operation to the remaining part of image $\hat{f}(x,y)$ outside the support is persuasive due to the known background grey-level.

This situation implies that the set theoretic strategy [4] would be more natural to utilize the *a priori* information over the support while the strategy of minimizing the distance between $\hat{f}(x,y)$ and background image is suitable to do outside the support.

Remark 2.1 Alternative straightforward set-theoretic strategy would be utilizing set-theoretically the *a priori* informations outside the support as well as its inside. However applying the well-known algorithms, for example POCS, in [7] for convex feasibility problems would fail to find the feasible solution to satisfy all requirements in general unless a restoration FIR filter has sufficiently large size, which will be shown through numerical examples in Section 5.

3. HYBRID STEEPEST DESCENT METHODS

Hybrid Steepest Descent Methods [5, 6] was recently developed to tackle a class of signal processing problems to be solved both in set theoretic as well as in optimal senses.

In this section, we briefly review *Hybrid Steepest Descent Methods* to the following optimization problem **(P)** (Convex optimization over generalized convex feasible set):

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Suppose that C_i $(i = 1, 2, \ldots, m)$ and K are nonempty closed convex sets and the function $\Phi : \mathcal{H} \to \mathbb{R}$ is defined by $\Phi(x) := \sum_{i=1}^m w_i d(x, C_i)^2$, $\sum_{i=1}^m w_i = 1$ and $w_i > 0$ for $i = 1, \ldots, m$. Let

$$K_\Phi := \{u \in K \mid \Phi(u) = \inf \Phi(K)\} \neq \emptyset$$

and

$$G := \begin{cases} \bigcap_{i=1}^m C_i & \text{if } \bigcap_{i=1}^m C_i \neq \emptyset, \\ K_\Phi & \text{if } \bigcap_{i=1}^m C_i = \emptyset. \end{cases}$$

Then, for a given continuous convex function $\Theta : \mathcal{H} \to \mathbb{R}$, the problem is

$$\text{Minimize } \Theta \text{ over } G.$$

We call K a *control set* and G a *generalized convex feasible set*. ¶

Remark 3.1 Note that the problem **(P1)** is not solvable by standard convex projection techniques[7] or nonlinear programming techniques[8, 9].

A mapping $T : \mathcal{H} \to \mathcal{H}$ is called **nonexpansive** if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in \mathcal{H}$. A **fixed point** of a mapping $T : \mathcal{H} \to \mathcal{H}$ is a point $x \in \mathcal{H}$ such that $T(x) = x$; the set of all fixed points of T is closed convex and denoted by $Fix(T)$. For any nonempty closed convex set $C \subset \mathcal{H}$, the mapping that assigns every point in $\mathcal{H}$ to its unique nearest point in C is called the **metric projection** onto C and is denoted by P_C. It is easy to see that $Fix(P_C) = C$ and to deduce that P_C is nonexpansive.

Definition 3.2 *Let S be a subset of a Hilbert space $\mathcal{H}$, and let a function $\Theta : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ be twice differentiable on some open set $U \supset S$. Then $\Theta'' : U \to B(\mathcal{H})$ is said to be* **uniformly strongly positive and uniformly bounded** *(or, briefly, Θ'' is USPUB)* **over** S *if $\Theta''(x)$ is self-adjoint for all $x \in S$, and there exist scalars $M \geq m > 0$ such that*

$$m\|v\|^2 \leq \langle \Theta''(x)v, v\rangle \leq M\|v\|^2 \text{ for all } x \in S \text{ and } v \in \mathcal{H}.$$

Example 3.3 *Suppose that $b \in \mathcal{H}$ and $A : \mathcal{H} \to \mathcal{H}$ is a strongly positive bounded linear operator, i.e., $\langle Ax, x\rangle \geq \alpha\|x\|^2$ for some $\alpha > 0$ and all $x \in \mathcal{H}$. Define a quadratic function $\Theta : \mathcal{H} \to \mathbb{R}$ by*

$$\Theta(u) := \frac{1}{2}\langle Au, u\rangle - \langle b, u\rangle \text{ for all } u \in \mathcal{H}.$$

Then $\Theta'' : \mathcal{H} \to B(\mathcal{H})$ satisfies the condition USPUB over $\mathcal{H}$.

Fact 3.4 *(Hybrid Steepest Descent Method (I)) Suppose that $T_i : \mathcal{H} \to \mathcal{H}$ $(i = 1, \ldots, N)$ are nonexpansive mappings with $F := \bigcap_{i=1}^N Fix(T_i) \neq \emptyset$ and*

$$\begin{aligned} F &= Fix(T_N \cdots T_1) = Fix(T_1 T_N \cdots T_3 T_2) \\ &= \ldots = Fix(T_{N-1} T_{N-2} \cdots T_1 T_N) \neq \emptyset, \end{aligned}$$

which is automatically satisfied for firmly nonexpansive mappings (more generally for attracting nonexpansive mappings) T_i's with $F \neq \emptyset$. Let $\Delta := \bigcup_{i=1}^N co(T_i(\mathcal{H}))$ and let a function $\Theta : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ be twice differentiable on some open set $U \supset \Delta$. Suppose that $\Theta'' : U \to B(\mathcal{H})$ satisfies the condition USPUB over Δ. Suppose that $(\lambda_n)_{n\geq 1}$ is a sequence of parameters in $[0, 1]$ that satisfies

$$\begin{aligned} &(B1)\ \lim_{n\to+\infty} \lambda_n = 0, \\ &(B2)\ \sum_{n\geq 1} \lambda_n = +\infty, \\ &(B3)\ \sum_{n\geq 1} |\lambda_n - \lambda_{n+N}| < +\infty. \end{aligned} \tag{3}$$

Then for an arbitrarily fixed μ with $0 < \mu < 2/M$ and any point $u_0 \in \mathcal{H}$, the sequence $(u_n)_{n\geq 0}$ generated by

$$\begin{aligned} u_{n+1} &:= T_{(n \bmod N)+1}(u_n) \\ &\quad -\lambda_{n+1}\mu\Theta'\left(T_{(n \bmod N)+1}(u_n)\right) \end{aligned}$$

converges to the unique minimizer u^* of the the function Θ over F.

The simplest example of oblivious sequences (λ_n) satisfying (3) may be $\lambda_n := \frac{1}{n}$ for $n = 1, 2, \ldots$.

When Θ is given as a quadratic function in Example 3.3, the problem (P1) is solved by the following scheme.

Corollary 3.5 *Let $K_\phi \neq \emptyset$, $\alpha \in (0, 3/2]$ and $\mu \in (0, 2/\|A\|)$. Assume that $(\lambda_n)_{n\geq 1}$ is a sequence of parameters satisfying (3) in $[0, 1]$ (for $N = 1$). Then for any point $u_0 \in \mathcal{H}$, the sequence $(u_n)_{n\geq 0}$ generated by*

$$\begin{aligned} u_{n+1} &:= \lambda_{n+1}\mu b \\ &+ (I - \lambda_{n+1}\mu A)\left\{\alpha P_K\left(\sum_{i=1}^m w_i P_{C_i}(u_n)\right) + (1-\alpha)u_n\right\} \end{aligned}$$

converges strongly to the unique minimizer of Θ over K_ϕ.

Remark 3.6 Note that all iterative algorithms introduced here are possible to employ any point in $\mathcal{H}$ as its starting point, which implies that all algorithms to find some approximate solutions, for example POCS[7] and other algorithms in [8, 9], can be used as a preprocessing of Hybrid Steepest Descent Method. Suitable preprocessing leads to great improvement of the convergence speed of Hybrid Steepest Descent Method.

4. PROPOSED BIND DECONVOLUTION SCHEME

As remarked in Section 2, certain mixture of set theoretic treatment as well as optimization is desired to solve the nonparametric blind deconvolution problem. In this section, we propose a simple set theoretic scheme to demonstrate how *Hybrid Steepest Descent Method* can be applied to the blind deconvolution problem.

Denote by $\{u(x, y)\}_{x=0,\ y=0}^{N_1-1, N_2-1}$ or $\boldsymbol{u} \in \mathbb{R}^{N_1\times N_2}$ the impulse response of a 2-D FIR restoration filter to be approximated to the inverse of PSF.

Define a collection of closed half spaces in a Euclid space $\mathbb{R}^{N_1\times N_2}$ by

$$\mathcal{C}_{1(x,y)} = \left\{\boldsymbol{u} \in \mathbb{R}^{N_1\times N_2} \mid (g * u)(x, y) \geq 0\right\},$$

for every pixel $(x, y) \in D_{sup}$. Obviously, $\bigcap_{(x,y)\in D_{sup}} \mathcal{C}_{1(x,y)}$ is the set of all FIR filters that output nonnegative values over the support D_{sup}.

Define also a hyperplane in $\mathbb{R}^{N_1\times N_2}$ by

$$\mathcal{C}_2 = \left\{\boldsymbol{u} \in \mathbb{R}^{N_1\times N_2} \mid \sum_{x=0}^{N_1-1}\sum_{y=0}^{N_2-1} u(x, y) = 1\right\}.$$

The set $\mathcal{C}_2$ is imposed due to the assumptions 5 and 6 in section 2. Then, the projections onto these sets are trivially given as follows. Projections onto $\mathcal{C}_{1(x,y)}$ and $\mathcal{C}_2$ are easily computed by

$$P_{1(x,y)}(\boldsymbol{u}) := \boldsymbol{u} - \frac{\langle \boldsymbol{g}_{xy}, \boldsymbol{u}\rangle}{\|\boldsymbol{g}_{xy}\|^2}\mathrm{cl}(\langle \boldsymbol{g}_{xy}, \boldsymbol{u}\rangle)\boldsymbol{g}_{xy},$$

and

$$P_2(\boldsymbol{u}) := \boldsymbol{u} + \frac{1 - \langle \boldsymbol{1}, \boldsymbol{u}\rangle}{\|\boldsymbol{1}\|^2}\boldsymbol{1},$$

where $\boldsymbol{g}_{xy} := (g(x,y), g(x, y-1), \ldots, g(x, y-N_2+1), \ldots, g(x-N_1+1, y-N_2+1))^t$ and $\mathbf{1} := (1,1,\ldots,1)^t \in \mathbb{R}^{N_1 \times N_2}$, where t denotes the transposition.

Our cost function $\Theta : \mathbb{R}^{N_1 \times N_2} \to \mathbb{R}$ is simply defined as a quadratic function

$$\Theta(\boldsymbol{u}) = \sum_{(x,y) \in \overline{D_{\text{sup}}}} [\hat{f}(x, y) - L_B]^2,$$

where we can assume by Example 3.3 that Θ satisfies US-PUB everywhere in $\mathbb{R}^{N_1 \times N_2}$ in almost practical situations[3].

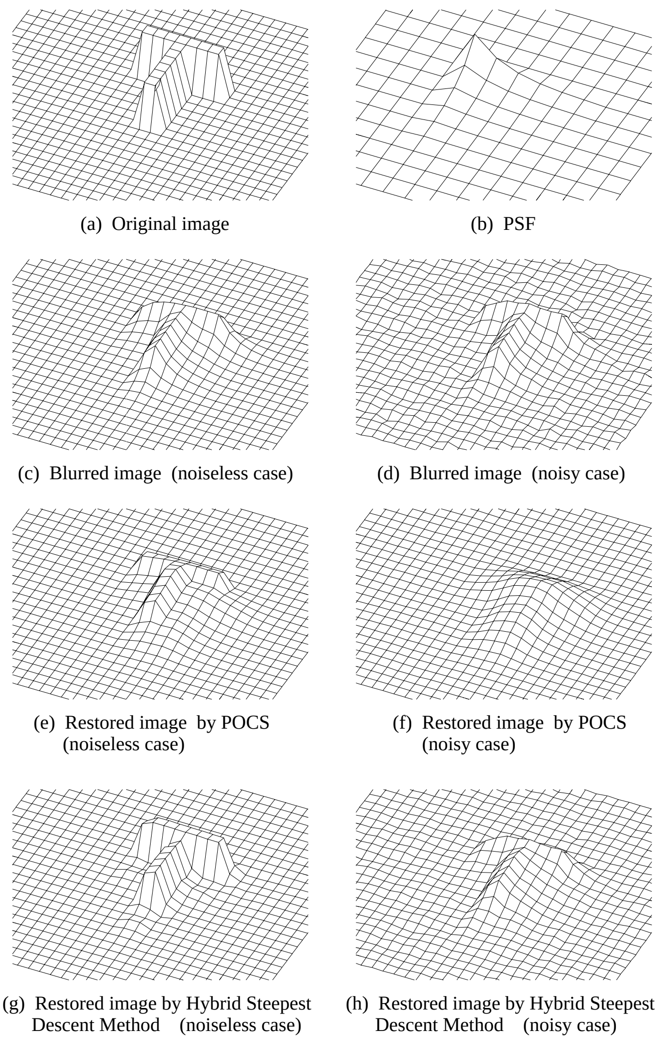
It is obvious that, for consistent case i.e. $\mathcal{C} := \bigcap_{(x,y) \in D_{\text{sup}}} \mathcal{C}_1(x, y) \cap \mathcal{C}_2 \neq \emptyset$, Fact 3.4 with $\mu := \frac{2}{Trace(\Theta'')}$, $\lambda_n := \frac{1}{n}$ and $\{T_i\} := \{P_{1(x,y)}\}_{(x,y) \in D_{sup}} \cup \{P_2\}$ can realize a simple set theoretic scheme to find the unique minimizer of Θ over $\mathcal{C}$.

5. NUMERICAL EXAMPLES

Though the following examples are only the cases where the condition $\mathcal{C} \neq \emptyset$ is satisfied, we can apply Corollary 3.5, in similar ways, to the inconsistent cases as well.

Suppose that (i) the size of the image to be restored is 30×30, (ii) the support size of the original object is 8×8, (iii) the Z-transform of PSF is sum of all monomials $\{h(m,n) z_1^m z_2^n\}_{0 \leq m \leq 5, 0 \leq n \leq 5}$ of Taylor series expansion of $\frac{1}{4}(1 - 0.5 z_1)^{-1}(1 - 0.5 z_2)^{-1}$, (iv) the background grey-level is $L_B = 0$, (v) For noisy case, noise is added at 40 dB BSNR, where $\text{BSNR} := 10 \log_{10} \left(\frac{\text{Blurred image power}}{\text{noise variance}} \right)$, and (vi) the size of the FIR restoration filter is 5×5 i.e. $(N_1, N_2) = (5, 5)$.

The examples shown in Fig. 2 suggest that the proposed scheme based on Fact 3.4 outperforms POCS in all cases, but its performance seems still strongly affected by additive noise. This situation should be improved by imposing additional *a priori* information such as the notion of *Total variation*[10], which will be presented elsewhere.



(a) Original image (b) PSF

(c) Blurred image (noiseless case) (d) Blurred image (noisy case)

(e) Restored image by POCS (noiseless case) (f) Restored image by POCS (noisy case)

(g) Restored image by Hybrid Steepest Descent Method (noiseless case) (h) Restored image by Hybrid Steepest Descent Method (noisy case)

Figure 2: Blind Deconvolution by Hybrid Steepest Descent Method and POCS

6. REFERENCES

[1] S. Haykin, ed., *Blind Deconvolution*, Prentice Hall, 1994.

[2] D. Kundur and D. Hatzinakos, "Blind Image Deconvolution", *IEEE Signal Processing Magazine*, pp. 43-63, vol. 13, no. 3, 1996.

[3] D. Kundur and D. Hatzinakos, "A Novel Blind Deconvolution Scheme for Image Restoration Using Recursive Filtering", *IEEE Trans. Signal Processing*, pp. 375-390, vol. 46, Feb, 1998.

[4] P. L. Combettes, "The foundations of set theoretic estimation", *Proc. IEEE*, 81, pp. 182-208, 1993.

[5] F. Deutsch and I. Yamada, "Minimizing certain convex functions over the intersection of the fixed point sets of nonexpansive mappings", *Numerical Functional Analysis and Optimization*, pp. 33-56, vol. 19, no. 1&2, 1998.

[6] I. Yamada, N. Ogura, Y. Yamashita and K. Sakaniwa, "Quadratic Optimization of Fixed Points of Nonexpansive Mappings in Hilbert Space", *Numerical Functional Analysis and Optimization*, pp. 165-190, vol. 19, no. 1&2, 1998.

[7] H. Stark and Y. Yang, *Vector Space Projections – A Numerical Approach to Signal and Image Processing, Neural Nets, and Optics*, John Wiley & Sons Inc, 1998.

[8] G. L. Nemhauser et al, *Optimization* – Handbooks in Operations Research and Management Science, vol. 1, North-Holland, 1989.

[9] J.-B. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Minimization Algorithms I, II*, Springer-Verlag, 1993.

[10] C. R. Vogel and M. E. Oman, "Fast, Robust Total Variation-Based Reconstruction of Noisy, Blurred Images", *IEEE Transactions on Image Processing*, vol. 7, no. 6, pp. 813-823, 1998.