NONPERIODIC SAMPLING AND RECONSTRUCTION FROM AVERAGES

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ABSTRACT

In this paper, we show that recovery of a function from its averages over squares in the plane is closely related to a problem of recovery of bandlimited functions from samples on unions of regular lattices. We use this observation to construct explicit solutions to the Bezout equation which can be easily implemented in software. We also show that these sampling results give a new proof of the "three squares theorem" which says that a function in the plane can be recovered from its averages on translates of three squares oriented parallel to the coordinate axes whose sidelengths are pairwise irrationally related. Other proofs of this theorem and construction of solution to the Bezout equation rely on interpolation methods in the theory of functions of one and several complex variables. Our sampling technique gives much simpler solutions especially in higher dimensions.

1. INTRODUCTION

The specific problem we address is the following. Given a collection of numbers $0 < r_1 < r_2 < \cdots < r_m$, can we find compactly supported distributions $\nu_1, \nu_2, \ldots, \nu_m$ on $\mathbf{R}^d$ which satisfy

$$\sum_{i=1}^{m} \mu_i * \nu_i = \delta, \tag{1}$$

where each μ_i is the characteristic function of a cube in $\mathbf{R}^d$? This problem will be referred to as the *multisensor deconvolution problem* or MDP.

The initial motivation for considering this problem is for its potential applications to signal and image recovery and enhancement [5]. In particular, solving (1) has the following interpretation. Given a d–dimensional signal or image, f, and its corresponding measured data $\{s_i\}_{i=1}^m$, where $s_i = f * \mu_i$, we can recover f from $\{s_i\}_{i=1}^m$ by

$$\sum_{i=1}^{m} s_i * \nu_i \;=\; \sum_{i=1}^{m} (f * \mu_i) * \nu_i = \sum_{i=1}^{m} f * (\mu_i * \nu_i)$$
$$= \; f * \sum_{i=1}^{m} \mu_i * \nu_i = f * \delta = f. \tag{2}$$

Clearly, physical limitations and numerical instabilities prevent the actual construction of an optical system with perfect reconstruction. However, it is believed that real increases in resolution can be obtained through the multisensor approach, that is, that it is possible to stably recover fine detail in the original signal f which is impossible to recover stably from each of the s_i.

In fact, if (1) can be solved with compactly supported deconvolvers, then recovery of f from $\{s_i\}_{i=1}^m$ is *local* in the following sense. Recovery of f at a point x_0 is possible from knowledge of each s_i on some compact set depending on i and containing x_0. Specifically, ($\widetilde{h}(x) = h(-x)$ here and below),

$$f(x_0) \;=\; \langle f, \tau_{x_0}\delta\rangle = \left\langle \tau_{-x_0} f,\; \sum_{i=1}^{m} \mu_i * \nu_i \right\rangle$$
$$= \; \sum_{i=1}^{m} \langle \tau_{-x_0} f * \mu_i, \nu_i\rangle = \sum_{i=1}^{m} \langle \tau_{-x_0} s_i, \nu_i\rangle. \tag{3}$$

Thus, $f(x_0)$ can be recovered from knowledge of s_i on the compact set $x_0 + \operatorname{supp} \nu_i$.

A theorem of Hörmander [10] asserts that equation (1) has a solution consisting of compactly supported distributions if and only if the following condition, known as the *strongly coprime condition* is satisfied. For some constants $A, B, N > 0$,

$$\sum_{i=1}^{m} |\widehat{\mu_i}(z)| \geq A(1 + |z|)^{-N} e^{-B|\Im z|} \tag{4}$$

for all $z \in \mathbf{C}^d$. If we require that the μ_i be characteristic functions of cubes in $\mathbf{R}^d$, i.e., $\mu_i = \chi_{[-r_i,r_i]^d}$, it follows from the work of Petersen and Meisters [11] that (4) is satisfied whenever $m \geq d+1$ and r_i/r_j is poorly approximated by rationals, that is, whenever there exist numbers $C, N > 0$ such that for all integers p, q, $|r_i/r_j - (p/q)| \geq C|q|^{-N}$.

While Hörmander's Theorem gives necessary and sufficient conditions under which compactly supported solutions to (1) exist, it gives no useful formula for constructing such solutions. This problem has been explored in a variety of contexts. See for example [4], [3], [5], and [6] as well as [7] for an expository overview.

2. SOLVING MDP WITHOUT COPRIMALITY

We can remove the assumption of coprimality and still solve the MDP in a slightly weaker sense. The following theorem is known as the "three squares theorem." It asserts that given three irrationally related positive numbers r_1, r_2, r_3, any function L^2 in the plane is completely determined by its averages on all squares of side r_1, r_2 or r_3 with sides parallel to the coordinate axes.

Theorem 1 ([9], [1]) *Let* $0 < r_1 < \cdots < r_m$, *let* $m = d + 1$. *Then the following are equivalent.*

(a) *The collection* $\{r_i\}_{i=1}^m$ *satisfies* r_i/r_j *is irrational for all* $i \neq j$.

(b) *If for* $1 \leq i \leq m$, $f \in L^2(\mathbf{R}^d)$ *satisfies*

$$f * \chi_{[-r_i,r_i]^d} \equiv 0$$

then $f \equiv 0$.

A local version of Theorem 1, known as the local three squares theorem, also holds. This theorem asserts that given three irrationally related positive numbers r_1, r_2, r_3, any function locally L^2 in the plane is completely determined on any square of side $R \geq r_1 + r_2 + r_3$ with sides parallel to the coordinate axes by its averages on all squares of side r_1, r_2 or r_3 with sides parallel to the coordinate axes which are completely contained within the larger square.

Theorem 2 ([9], [2]) *Let* $0 < r_1 < \cdots < r_m$, *let* $m = d + 1$, *and let* $R = \sum_{i=1}^m r_i$. *Then the following are equivalent.*

(a) *The collection* $\{r_i\}_{i=1}^m$ *satisfies* r_i/r_j *is irrational for* $i \neq j$.

(b) *If for* $1 \leq i \leq m$, $f \in L^2[-R, R]^d$ *satisfies*

$$f * \chi_{[-r_i,r_i]^d} = 0 \text{ on } [-R + r_i, R - r_i]^d$$

then $f \equiv 0$.

Note that in Theorems 1 and 2, the only assumption made about the sidelengths of the cubes is that they are irrationally related. The strongly coprime assumption is not made. The cost is that we know only that the function is completely determined and we have no explicit formula such as (3) for its recovery.

It will be the goal of this paper to show that sampling theory can be used very effectively to recover a function locally from its averages and that solutions in higher dimensions are essentially no more complicated that one dimensional solutions.

3. SAMPLING AND THE MDP

It has been well established in several papers [12], [7], [13], [14], [9], [8] that the theory of sampling of bandlimited functions and solving convolution equations such as (1) are closely related. In this section, we will give a brief summary of this connection and refer the reader to the literature for more details.

Consider the case $d = 1$. Taking the Fourier transform of (1) gives

$$\sum_{i=1}^{m} \widehat{\mu_i}(\xi)\widehat{\nu_i}(\xi) = 1. \tag{5}$$

Let σ be a permutation of the set $\{1, 2, \ldots, m\}$ with the property that $\sigma(i) \neq i$ for all i. We seek solutions to (1) of the form

$$\widehat{\nu_i}(\gamma) = \widehat{f_{\sigma(i)}}(\gamma) \prod_{j \neq i, \sigma(i)} \widehat{\mu_j}(\gamma). \tag{6}$$

Substituting (6) into (5) and rearranging terms gives

$$\sum_{i=1}^{m} \widehat{f_i}(\gamma) \prod_{j \neq i} \widehat{\mu_j}(\gamma) = 1. \tag{7}$$

Solving (5) and hence (1) is equivalent to finding distributions f_i which satisfy (7). Note also that any solutions $\{\nu_i\}_{i=1}^m$ obtained in this way have the form $\nu_i = f_{\sigma(i)} * p_i$ where p_i is an $(m-2)$–fold convolution product of characteristic functions of intervals.

If we let Λ_k be the zero set of $\widehat{\mu_k}$ then clearly

$$\Lambda_k = \{n/2r_k : n \in \mathbf{Z} \setminus \{0\}\}. \tag{8}$$

It is also clear that if $\lambda \in \Lambda_k$ then equation (7) reduces to

$$\widehat{f_k}(\lambda) \prod_{j \neq k} \widehat{\mu_j}(\lambda) = 1$$

since the product vanishes whenever $i \neq k$. Thus, for $\lambda \in \Lambda_k$, f_k must satisfy

$$\widehat{f_k}(\lambda) = \left(\prod_{j \neq k} \widehat{\mu_j}(\lambda) \right)^{-1}.$$

Because the ratios r_i/r_j are irrational, $\widehat{f_k}(\lambda)$ has polynomial growth in λ. We would like to assert that

$$f_k = \left(c_0^k + \sum_{\lambda \in \Lambda_k} \widehat{f_k}(\lambda)\, e^{2\pi i \lambda x} \right) \chi_{[-r_k, r_k]}$$

for some appropriate choice of c_0^k but because of the polynomial growth of $\widehat{f_k}(\lambda)$ it is not clear that this sum and product make sense.

In order to make sense of the above sum, we choose a function $\varphi \in C_c^\infty(\mathbf{R})$ with support in the interval $[-R, R]$ where $R = \sum_{i=1}^m r_i$ and define

$$f_{k,\varphi} = \left(\widehat{\varphi}(0)c_0^k + \sum_{\lambda\in\Lambda_k} \widehat{\varphi}(\lambda)\widehat{f_k}(\lambda)\, e^{2\pi i\lambda x}\right)\chi_{[-r_k,r_k]} \quad (9)$$

which is a well–defined function. With $\nu_{i,\varphi} = f_{\sigma(i),\varphi} * p_i$, and with the constants $\{c_0^k\}_{k=1}^m$ chosen appropriately, it follows that the equation

$$\sum_{i=1}^m \widehat{\mu_i}(\gamma)\widehat{\nu_{i,\varphi}}(\gamma) = \widehat{\varphi}(\gamma) \quad (10)$$

is satisfied on the set $\Lambda = \cup_{i=1}^m \Lambda_i \cup \{0\}$. It is easy to see that both sides of (10) are the Fourier transforms of C^{m-2} functions supported in the interval $[-R, R]$. Thus, the question of whether $\{\nu_{i,\varphi}\}_{i=1}^m$ satsifies (10) for all γ reduces to the question of whether a C^{m-2} function supported on $[-R, R]$ is completely determined by the samples of its Fourier transform on Λ. The answer to this question is given in the following theorem.

Theorem 3 (cf. [13], Theorem 3.2) *Let $0 < r_1 < r_2 < \cdots < r_m$ be such that $m \geq 2$, and r_i/r_j is irrational if $i \neq j$, let Λ_i be given by (8) and let $R = \sum_{i=1}^m r_i$. If $F \in C^{m-2}(\mathbf{R})$ with $\operatorname{supp} F \subseteq [-R, R]$ satisfies $\widehat{F}(\lambda) = 0$ for all $\lambda \in \cup_{i=1}^m \Lambda_i \cup \{0\}$. Then $F(x) = 0$ on $[-R, R]$.*

Finally, if we take φ to be the elements of an approximate identity, and let $\varphi \to \delta$ (as a distribution), then it is possible to show that the limits $\lim_{\varphi\to\delta} f_{k,\varphi}$ make sense as distributions and in fact converge to f_k for each k. Consequently, it can be shown that the functions $\nu_{i,\varphi}$ converge to well–defined, compactly supported distributions ν_i which satisfy (1). Hence, sampling theory can be used to find explicit solutions to (1).

4. SOLUTIONS TO THE MDP WITHOUT COPRIMALITY

It is instructive to consider where the assumption of coprimality was used in the previous section. A few momment reflection reveals that the assumption of coprimality guarantees that the coefficients

$$\widehat{f_k}(\lambda) = \left(\prod_{j\neq k} \widehat{\mu_j}(\lambda)\right)^{-1}$$

have polynomial growth in λ. This fact guarantees that the coefficients $f_{k,\varphi}(\lambda) = \widehat{\varphi}(\lambda)\widehat{f_k}(\lambda)$ decay rapidly since the numbers $\widehat{\varphi}(\lambda)$ decay faster than any polynomial in λ. Therefore, we can conclude that the sum in (9) converges and that $f_{k,\varphi}$ is well–defined.

A different way to force convergence in (9) is to consider functions φ for which $\widehat{\varphi}(\lambda) = 0$ for all but finitely many $\lambda \in \Lambda$. In this case, the sums defining $f_{k,\varphi}$ will converge regardless of the growth at infinity of the coefficients $\widehat{f_k}(\lambda)$, and hence the strongly coprime condition can be ignored. In this case, the inner products $\langle f, \varphi\rangle$ can be computed via

$$\langle f, \varphi\rangle = f * \widetilde{\varphi}(0) = \sum_{i=1}^m (f * \mu_i) * \widetilde{\nu_{i,\varphi}}(0) = \sum_{i=1}^m \langle s_i, \nu_{i,\varphi}\rangle. \quad (11)$$

If the set of φ whose Fourier transforms vanish on all but finitely many points of Λ form a complete set, then f is completely determined by the data $\{s_i\}_{i=1}^m$. This suggests a strategy for solving the MDP without coprimality.

We start with the following result which was proved in [13] and which is closely related to Theorem 3. The remainder of this section is taken essentially from [9].

Theorem 4 ([13], Theorem 3.1) *Let $0 < r_1 < r_2 < \cdots < r_m$ be such that r_i/r_j is irrational if $i \neq j$, let Λ_i be given by (8) and let $R = \sum_{i=1}^m r_i$. Suppose that $F \in L^2[-R, R]$ satisfies,*

(a) $\widehat{F}(\lambda) = 0$, *for all* $\lambda \in \cup_{i=1}^m \Lambda_i$,

(b) $\widehat{F}^{(j)}(0) = 0$, *for* $j = 0, 1, \ldots, m-1$.

Then $F(x) = 0$ a.e. on $[-R, R]$.

Equivalently, the collection

$$\Lambda^* = \{e^{2\pi i\lambda x}\}_{\lambda\in\cup_{i=1}^m \Lambda_i} \cup \{1, x, \ldots, x^{m-1}\}$$

is complete in $L^2[-R, R]$.

It turns out that the set Λ^* is *minimal* in $L^2[-R, R]$, that is, it has a biorthogonal sequence. To see why this is true, define for $j = 0, \ldots, m-1$ the functions $g_j \in L^2(\widehat{\mathbf{R}})$ by

$$g_j(x) = \frac{t^j}{j!} \prod_{k=1}^m \frac{\sin(2\pi r_k x)}{2\pi r_k x}. \quad (12)$$

Let $f_{0,m-1} = g_{m-1}$ and define $f_{0,j}$ recursively for $j = m-2$ down to $j = 0$ by

$$f_{0,j}(x) = g_j(x) + \sum_{\ell=j+1}^{m-1} \left(\frac{d^\ell}{dx^\ell} g_j(0)\right) f_{0,\ell}(x). \quad (13)$$

For $\lambda = n/2r_j \in \Lambda_j$, define $g_\lambda \in L^2(\widehat{\mathbf{R}})$ by

$$g_\lambda(x) = \frac{\sin(2\pi r_j(x-\lambda))}{2\pi r_j(x-\lambda)} \frac{\prod_{k\neq j} \frac{\sin(2\pi r_k x)}{2\pi r_k x}}{\prod_{k\neq j} \frac{\sin(2\pi r_k \lambda)}{2\pi r_k \lambda}} \quad (14)$$

and define f_λ by

$$f_\lambda(x) = g_\lambda(x) - \sum_{\ell=0}^{m-1} \left(\frac{d^\ell}{dx^\ell} g_\lambda(0) \right) f_{0,\ell}(x). \qquad (15)$$

The following theorem holds.

Theorem 5 (cf. [13], Proposition 3.1) *Let* $\widehat{F_{0,j}} = f_{0,j}$ *and* $\widehat{F_\lambda} = f_\lambda$ *and define*

$$\mathcal{F} = \{F_\lambda : \lambda \in \cup_{i=1}^m \Lambda_i\} \cup \{F_{0,j} : j = 0, \ldots, m-1\}.$$

Then $\mathcal{F} \subseteq L^2[-R, R]$ *and* $\mathcal{F}$ *is biorthogonal to* Λ^*.

Theorem 5 follows from the observation that (a) $f_{0,j}(\lambda) = 0$ for $0 \le j \le m-1$ and $\lambda \in \Lambda$, and $\frac{d^k}{dt^k} f_{0,j}(0) = \delta_{j,k}$ for $0 \le j,\ k \le m-1$ and (b) $f_\lambda(\lambda') = \delta_{\lambda,\lambda'}$ for $\lambda,\ \lambda' \in \Lambda$, and $\frac{d^k}{dt^k} f_\lambda(0) = 0$ for $0 \le k \le m-1$.

Thus, for every $\varphi \in \mathcal{F}$, $f_{k,\varphi}$ defined by (9) is well–defined and supported in $[-r_k, r_k]$. Consequently, the functions $\nu_{i,\varphi}$ are well–defined functions supported in the interval $[-R + r_i, R - r_i]$. It follows that calculating $\langle f, \varphi \rangle$ by (11) requires knowledge of s_i only on $[-R + r_i, R - r_i]$. It remains to show that in fact a function $f \in L^2[-R, R]$ is completely determined by the inner products $\langle f, \varphi \rangle$ with $\varphi \in \mathcal{F}$. That this is true is proved in [9] and relies heavily on a result of Young [15] which states that a set biorthogonal to an *exact sequence of exponentials* is always complete on $L^2(-\pi, \pi)$.

Combining the above observations we arrive at a proof of one direction of Theorem 2 in one dimension. Generalizing to higher dimensions is essentially trivial and relies on the observation that d–fold tensor products of complete sets in $L^2[-R, R]$ are complete sets in $L^2[-R, R]^d$. The other direction is also not difficult and details can be found in [9].

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