NON-UNIFORM SAMPLING IN WAVELET SUBSPACES

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ABSTRACT

It is well known that the Shannon sampling theorem can be put into a wavelet context. But it has also been shown that for most wavelets, a sampling theorem for the associated subspaces exists. There is even a non-uniform sampling theorem as in the Shannon case. In general the bounds on the deviations from uniform are not as easy to specify in this case. No simple Kadec 1/4 theorem holds except in special cases (such as the Franklin where the bound is 1/2). For a particular class, the Meyer wavelets, which are bandlimited but with a smooth spectrum, a similar bound is sometimes obtainable. Unfortunately, it is much smaller that 1/4.

I.. THE SHANNON THEOREM.

The Shannon sampling theorem, by which a $\pi-$band limited analog signal may be reconstructed from its sampled values, refers to the formula

$$f(t) = \sum_n f(n)s(t-n) \tag{1}$$

where s refers to the sinc function $s(t) = (\sin \pi t)/\pi t$ [5]. This may be modified to allow irregular sampling and still recover the signal,

$$f(t) = \sum_n f(t_{n)}\xi_n(t) \tag{2}$$

provided $\sup_n |t_n - n| < 1/4$. This is the Kadec 1/4 Theorem [8].

II.. WAVELET SAMPLING.

The requirement that the signal be band limited can be avoided by considering signals in other wavelet subspaces. (The spaces of $2^m\pi$ band limited signals are such a subspace). For most father wavelets $\varphi(t)$ and their associated multiresolution ladder of subspaces $\{\mathrm{V}_m\}$, there is sampling theorem for signals in V_0 which has the same form as (1) except that $S(t)$ is no longer the sinc function [6]. Rather it is defined in terms of its Fourier transform by $\widehat{S(\omega)} = \widehat{\varphi(\omega)}/\widehat{\varphi * (\omega)}$, where

$$\widehat{\varphi * (\omega)} = \sum_n \varphi(n)e^{-in\omega}. \tag{3}$$

Of course $\widehat{\varphi} * (\omega)$ may not have any zeros, a condition that holds for most father wavelets.

This enables us to get regular sampling theorems even for some time limited signals as well as other theorems for band limited signals. However the extension to irregular sampling is not so straightforward. For shifted sampling, i.e., sampling at non-integer shifts of the integers, some more general results are possible by replacing (3) by the requirement that

$$\widehat{\varphi_\alpha * (\omega)} = \sum_n \varphi(\alpha+n)e^{-in\omega} \neq 0 \tag{4}$$

for some real number α.This gives us a sampling theorem [2].

$$f(t) = \sum_n f(n+\alpha)\xi_{\alpha,n}(t) \tag{5}$$

after similar calculations. This was extended in [3] to a further shifted sampling theorem. Under the same hypothesis, there is $\delta_0 > 0$ such that

$$f(t) = \sum_n f(n+\alpha+\delta)\xi_{\alpha,n}(t) \qquad (6)$$

for $|\delta| \leq \delta_0$.

An irregular sampling theorem in [3] was sharpened in [1] to obtain a theorem whose conclusion is of the form

$$f(t) = \sum_n f(n+\alpha+\delta_n)\xi_{\alpha,n,\delta}(t) \qquad (7)$$

for $|\delta_n| \leq \delta$.

In most cases the δ which appears in (7) cannot be calculated explicitly. However for the Franklin wavelet, in which the scaling function is obtained by orthogonalization of the "hat function", stronger results are possible.

Proposition 1. Let V_0 be the wavelet subspace of the Franklin wavelets, $|\delta_k| \leq \delta < \frac{1}{2}$; then there is a sampling sequence $\{\xi_k(t)\} \subseteq V_0$ such that

$$f(t) = \sum_k f(k+\delta_k)\xi_k(t) \qquad (8)$$

for all $f \in V_0$.

The proof uses the fact that for the reproducing kernel $k(t,a)$ of $V_0, \{k(t,t_k)\}$ is a frame in V_0 (see[3]).

III.. MEYER WAVELETS.

The previous results, except the last, apply to most standard wavelets. We can get more precise results by considering a particular, the Meyer wavelets, which consist of band limited functions. Their father wavelet, $\varphi(t)$, has properties similar the the sinc function, but is more rapidly decreasing than it as t$\rightarrow \pm\infty$. The Fourier transform has the property that $\widehat{\varphi}(\omega)$ has its support in $[-\pi-\varepsilon,\pi+\varepsilon]$, for some $0<\varepsilon<\pi/3$, that is has a slightly larger bandwidth than the sinc function. However it can be chosen to be as smooth as we wish even C^∞.

A particular example is given by the raised cosine wavelet [7] in which the orthogonal father wavelet is given by

$$\begin{aligned}\varphi(t) &= \frac{\sin\pi(1-\beta)t + 4\beta t\cos\pi(1+\beta)t}{\pi t(1-(4\beta t)^2)} \qquad (9)\\ 0 &< \beta < 1/3.\end{aligned}$$

Many non-orthogonal sampling functions in closed form are possible to find, e.g.,[9]

$$\xi(t) = \frac{\sin\pi t\cos\pi\beta t}{\pi t(1-4(\beta t)^2)}, 0<\beta<1/3. \qquad (10)$$

Another is given by the Bessel functions

$$\xi(t) = \frac{2^\nu\Gamma(\nu+1)\sin\pi t J_\nu(\pi\beta t)}{\pi t(\pi\beta t)^\nu}. \qquad (11)$$

These give uniform sampling theorems for $f \in V_0$ which in this case is the closed linear span of $\{\xi(t-n)\}$ which also constitutes a Riesz basis. For non-uniform sampling we have

Theorem 1 . Let $\varphi(t)$ be a symmetric father wavelet of Meyer type; let $|t_n - n| \leq 0.04$. Then there is a sequence $\{\xi_n(t)\} \subseteq V_0$ such that for all $f \in V_0$,

$$f(t) = \sum_k f(t_k)\xi_k(t).$$

The proof involves showing that $\{q(t,t_n)\}$, where $q(t,u)$ is the reproducing kernel of V_0, is a Riesz basis of V_0. This in turn is done by finding the Fourier transform $\widehat{q}(\omega,t_n)$ and showing that it is a Riesz basis of the Fourier transform of V_0 by comparing it to $\widehat{q}(\omega,n)$ which is known to be such a basis. Then we find that

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}|\sum_n c_n[\widehat{q}(\omega,t_n)-\widehat{q}(\omega,n)]|^2 d\omega\\ \leq\ &\gamma\sum_n|c_n|^2\end{aligned}$$

and

$$\frac{1}{\sqrt{2}}\sum_n |c_n|^2 \le \frac{1}{2\pi}\int_{-\infty}^{\infty} |\sum_n c_n \widehat{q}(\omega,n)]|^2 d\omega.$$

We also use the result [8,p.44]

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} |\sum_n c_n[e^{in\omega}-e^{it_n\omega}]|^2 d\omega \\ \le\ & [1+\sqrt{2}\sin\pi(L-\frac{1}{4})]^2\sum_n |c_n|^2. \end{aligned}$$

By choosing t_n as in the hypothesis $\sqrt{2}\gamma$ can be made less than 1. This is the condition needed for a Riesz basis [8,p.38].

Once we have the Riesz basis $\{q(t,t_n)\}$,we use the dual biorthogonal basis [8,p.32] to get our sampling functions $\{\xi_n(t)\}$. Then the expansion of $f\in V_0$ is

$$f(t)=\sum_k\int_{-\infty}^{\infty} f(u)q(u,t_k)du\xi_k(t).$$

which gives us the result we want since $q(u,t)$ is the reproducing kernel.

Since the Meyer wavelets are band limited, we can get an alternate sampling theorem for band limited signals. This requires a narrower bandwidth ($\pi-\varepsilon$ instead of π), but has the advantage of better time localization and more rapid convergence of the series.

The wavelets approach has another advantage. The aliasing error may be given in terms of coefficients of the mother wavelet. This allows an alternate systematic approach to aliasing [6].

IV.. REFERENCES

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