A MULTIDIMENSIONAL IRREGULAR SAMPLING ALGORITHM AND APPLICATIONS

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ABSTRACT

For a given spiral, a bandwidth B can be chosen and a sequence S can be constructed on the spiral with the property that all finite energy signals having bandwidth B can be reconstructed from sampled values on S. The bandwidth can be expanded as desired, and reconstruction is attained by constructing sampling sets on interleaving spirals. This solves a problem in MRI; and the algorithm can be modified to deal with irregular sampling problems in SAR. The algorithm is a consequence of our theoretical results, which in turn were inspired by seminal work on balayage in the 1960s by Beurling and Landau. Our results depend on d-dimensional Fourier frames and tiling properties of spectral synthesis sets.

1. INTRODUCTION

We shall present a d-dimensional reconstruction algorithm for given irregularly spaced data. The algorithm is based on our theorem, which provides sufficient conditions in order that a discrete subset $\Lambda \subseteq \hat{\mathbf{R}}^d$ should determine a Fourier frame for the space $L^2(E)$ of finite energy signals on $E \subseteq \mathbf{R}^d$. $\hat{\mathbf{R}}^d$ is Euclidean space $\mathbf{R}^d$, but is meant to denote the Fourier domain of the space signals defined on $\mathbf{R}^d$. The sufficient conditions include a fundamental tiling property and the fact that E must satisfy spectral synthesis, e.g., [1], Our approach depends on and is inspired by the deep work of Beurling [2] and Landau [4], [5] dealing with balayage and with sampling and interpolation sets.

The algorithm is naturally applicable in some basic problems in MRI and SAR. For example, in MRI it is important to reconstruct signals on $\mathbf{R}^d$ for discrete spectral data on interleaving spirals, e.g., [9]. Spiral imaging is used because it provides a setting for fast imaging methods. In our algorithm, we construct discrete subsets Λ of a spiral A (for Archimedean spiral), and then we are able to reconstruct the elements of $L^2(E)$ in terms of the Fourier frame determined by Λ. The domain E can be made as large as desired by choosing interleaving spirals. Similarly, our algorithm can be used in spotlight mode synthetic-aperture radar, where Fourier domain data is available on polar grids contained in small annular wedges, e.g., [6], [7].

Section 2 gives the results from the theory of frames that are required to formulate our theorem. Our theorem is presented in Section3. In Section 4 we show how to reformulate our theorem as a constructive algorithm in the case of the aforementioned signal reconstruction for given spectral data on spirals. In order to implement the algorithm it is important to have useful frame bounds, and these are given in Section 5.

2. FRAMES

Definition 1 *Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\{\, x_n : n \in \mathbf{Z}^d \,\} \subseteq \mathcal{H}$ is a* frame *if there exist $0 < A \leq B < \infty$ such that for all y in $\mathcal{H}$*

$$A\,\|y\|^2 \;\leq\; \sum |\langle y, x_n\rangle|^2 \;\leq\; B\,\|y\|^2.$$

A and B are frame bounds. *If $A = B$, then the frame is a* tight frame. *If $\{x_n\}$ is no longer a frame when we delete any element from it, then $\{x_n\}$ is an* exact frame.

Definition 2 *Let $\{x_n\}$ be a frame for $\mathcal{H}$. The frame operator $S : \mathcal{H} \longrightarrow \mathcal{H}$ is defined by*

$$\forall\, y \in \mathcal{H}, \qquad S\,y \;=\; \sum \langle y, x_n\rangle\, x_n.$$

Theorem 3 *Let $\{x_n\}$ be a frame for $\mathcal{H}$ with frame bounds A and B, then*

(a) $AI \leq S \leq BI$, where $I : \mathcal{H} \longrightarrow \mathcal{H}$ is the identity mapping. In particular, S is positive, and therefore self-adjoint.

(b) S is invertible, and $B^{-1}\, I \leq S^{-1} \leq A^{-1}\, I$.

(c) $\{S^{-1}\, x_n\}$ is a frame with frame bounds B^{-1} and A^{-1}. It is called the dual frame *of $\{x_n\}$.*

(d) For every $y \in \mathcal{H}$

$$y \;=\; \sum \langle y,\, S^{-1}\, x_n\rangle\, x_n \;=\; \sum \langle y,\, x_n\rangle\, S^{-1} x_n. \tag{1}$$

Remark 4 *If $\{x_n\}$ is a frame for $\mathcal{H}$ with frame bounds A, B and frame operator S, then it is easy to see that*

$$\|I - \frac{2S}{A+B}\| \leq \frac{B-A}{A+B} < 1, \tag{2}$$

where $I : \mathcal{H} \longrightarrow \mathcal{H}$ is the identity operator. The inequality (2) allows us to prove that

$$S^{-1} = \frac{2}{A+B} \sum_{k=0}^{\infty} \left(I - \frac{2S}{A+B} \right)^k.$$

This is the most elementary approach to implement frames for signal reconstruction, and it is meant to illustrate the importance of effective frame bounds, see Section 5.

Example 5 *Let $\Lambda \subset \hat{\mathbf{R}}^d$ be a sequence and let $E \subset \mathbf{R}^d$ have finite Lebesgue measure. By the Parseval Formula, the following are equivalent.*

(1) $\{ e_{-\lambda} : \lambda \in \Lambda \}$ is a frame for $L^2(E)$.

(2) There exist $0 < A \leq B < \infty$ such that

$$A \, \|\phi\|_2^2 \leq \sum_{\lambda \in \Lambda} |\phi(\lambda)|^2 \leq B \, \|\phi\|_2^2, \tag{3}$$

for all ϕ in the Paley-Wiener space PW_E.

For convenience, in the case of (3), we say that Λ is a Fourier frame *for PW_E.*

3. THE TILING THEOREM

Duffin and Schaeffer [3] proved the following theorem:

Theorem 6 *Let $d, L, \delta > 0$ and let $\{\lambda_n\} \subseteq \mathbf{R}$ satisfy the properties that $\{\lambda_n\}$ is uniformly dense, i.e., $|\lambda_n - n/d| \leq L$ for all n, and $\{\lambda_n\}$ is uniformly discrete, i.e., $|\lambda_n - \lambda_m| \geq \delta$ when $n \neq m$. Then $\{\lambda_n\}$ is a Fourier frame for PW_E, where $E = [-r/2, r/2]$ and $0 < r < d$.*

It is easy to extend Duffin and Schaeffer's result to higher dimensions in the following way.

Theorem 7 *We say a sequence $\{\lambda_n\}$ satisfies the condition $UD(d, L, \delta)$, if it satisfies the following inequalities:*

$$\begin{cases} |\lambda_n - \frac{n}{d}| \leq L, & n \in \mathbf{Z} \\ |\lambda_n - \lambda_m| \geq \delta, & n \neq m. \end{cases} \tag{4}$$

In $\hat{\mathbf{R}}^2$, if Λ is of the form $\{ (\lambda_{mn}, \gamma_n) : m, n \in \mathbf{Z} \}$ and there exist $d_1, L, \delta > 0$ such that for fixed n, $\tau_{\lambda_{0n}} \{\lambda_{mn}\}$ satisfies the condition $UD(d_1, L, \delta)$, and $\{\gamma_n\}$ has uniform density d_2, then X is a Fourier frame for $PW_{[-r_1/2, r_1/2] \times [-r_2/2, r_2/2]}$ whenever $r_1 < d_1$ and $r_2 < d_2$.

Unfortunately, this result only provides one-dimensional freedom.

Let $E \subseteq \mathbf{R}^d$ be a convex, compact set which is symmetric about the origin and has non-empty interior. Then $\|\cdot\|_E$, defined by

$$\forall x \in \mathbf{R}^d, \ \|x\|_E = \inf\{r > 0 : x \in rE\},$$

is a norm on $\mathbf{R}^d$ equivalent to the Euclidean norm. The polar set $E^* \subseteq \hat{\mathbf{R}}^d$ of E is defined by

$$E^* = \{ \gamma \in \hat{\mathbf{R}}^d : x \cdot \gamma \leq 1, \ \text{for all } x \in E \}.$$

Obviously, E^* is a convex, compact set which is symmetric about the origin and has non-empty interior.

Example 8 *(a) Let $E = [-1, 1] \times [-1, 1]$. Then for $(x_1, x_2) \in \mathbf{R}^2$,*

$$\begin{aligned} \|(x_1, x_2)\|_E &= \inf \{ r : |x_1| \leq r, |x_2| \leq r \} \\ &= \|(x_1, x_2)\|_\infty. \end{aligned}$$

And the polar set of E is

$$\begin{aligned} E^* &= \{ (\gamma_1, \gamma_2) : |\gamma_1| + |\gamma_2| \leq 1 \} \\ &= \{ (\gamma_1, \gamma_2) : \|(\gamma_1, \gamma_2)\|_1 \leq 1 \}. \end{aligned}$$

(b) For $p > 1$, let $E = \{ (x_1, x_2) : \|(x_1, x_2)\|_p \leq 1 \}$. Then $E^* = \{ (\gamma_1, \gamma_2) : \|(\gamma_1, \gamma_2)\|_q \leq 1 \}$, where $1/p + 1/q = 1$.

Theorem 9 *Let $E \subseteq \mathbf{R}^d$ be a convex, compact set which is symmetric about the origin and has non-empty interior, and let $\Lambda \subseteq \hat{\mathbf{R}}^d$ be a uniformly discrete set satisfying the tiling property*

$$\bigcup_{\lambda \in \Lambda} \tau_\lambda \, E^* = \hat{\mathbf{R}}^d.$$

If $r < 1/4$, then Λ is a Fourier frame for PW_{rE}.

Our proof involves the Paley-Wiener Theorem and properties of balayage, and it depends on the profound work in [2] and [4]. The following example shows that $1/4$ is the best possible.

Example 10 *Let*

$$\lambda_{m,n} = \begin{cases} (m + 1/2, n - 1) & \textit{if } m \textit{ is odd, } n \textit{ is even,} \\ (m, n) & \textit{otherwise.} \end{cases} \tag{5}$$

Note that

$$\bigcup \tau_{\lambda_{m,n}} \left(\{ (x, y) : \|(x, y)\|_1 \leq 1 \} \right) = \mathbf{R}^2.$$

It is not difficult to show that $\{\lambda_{m,n}\}$ can not be a Fourier frame for $PW_{[-r/2, r/2]^2}$ whenever $r > 1/2$.

4. EXAMPLES OF IRREGULAR SAMPLING RECONSTRUCTION

We shall use the results from Section 3 to give a constructive irregular sampling signal reconstruction method for the case of interleaving spirals. Example 11 shows how to construct Λ on a spiral in $\hat{\mathbf{R}}^2$ to obtain a Fourier frame for some PW_E. Example 12 shows how interleaving spirals are required to do signal reconstruction for functions having a given bandwidth. Thus, if we are given a bandwidth E and a finite union A of sufficiently many rotations of a given spiral, we can construct a Fourier frame $\Lambda \subseteq A$ for PW_E.

Example 11 *Fix $c > 0$. For any given $r > 0$ with $cr < 1/2$, we shall show how to choose a uniformly discrete subset Λ_r of the spiral*

$$A_c = \{ (c\gamma \cos 2\pi\gamma, c\gamma \sin 2\pi\gamma) : \gamma \geq 0 \} \subseteq \hat{\mathbf{R}}^2$$

such that Λ_r is a Fourier frame for $PW_{\bar{B}(0,r)}$.

Let $(\eta_0, \xi_0) = \gamma_0(\cos 2\pi\theta_0, \sin 2\pi\theta_0) \in \mathbf{R}^2$. We have

$$\begin{aligned} &dist((\eta_0,\xi_0), A_c) \\ &\qquad \leq dist(\gamma_0, \gamma_0 + \{ c(n+\theta_0) : n \in \mathbf{N} \cup \{0\} \}) \leq c/2. \end{aligned}$$

In fact, $\sup_{(\eta_0,\xi_0)\in\mathbf{R}^2} dist((\eta_0,\xi_0), A_c) = c/2$.

Now, take $\delta > 0$ such that $(\delta + c/2)r < 1/4$. We choose the set of points Λ_r along the spiral having curve distance between consecutive points less than 2δ. Then the distance from any point on the spiral A_c to Λ_r is less than δ. Further, the distance from any point in $\mathbf{R}^2$ to the spiral A_c is less than $c/2$. Thus, by the triangle inequality, the distance from any point in $\mathbf{R}^2$ to Λ_r is less than $(c/2 + \delta)$. This implies that Λ_r is a Fourier frame for $PW_{\bar{B}(0,r)}$.

Example 12 *For any $r > 0$, we shall show how to choose a uniformly discrete subset Λ_r of a finite union of rotations of the spiral $A = \{ (\gamma \cos 2\pi\gamma, \gamma \sin 2\pi\gamma) : \gamma \geq 0 \}$ such that Λ_r is a Fourier frame for $PW_{\bar{B}(0,r)}$.*

First, choose $M \in \mathbf{N}$ such that $\gamma/M < 1/2$, and define

$$A = \bigcup_{0}^{M-1} A_k,$$

where

$$A_k = \{ (\gamma \cos 2\pi(\gamma - \frac{k}{M}), \gamma \sin 2\pi(\gamma - \frac{k}{M})) : \gamma \geq 0 \}.$$

Now, given any point (η_0, ξ_0) in $\hat{\mathbf{R}}^2$, there exist $\gamma_0 \geq 0, \theta_0 \in [0,1)$ such that

$$(\eta_0, \xi_0) = (\gamma_0 \cos 2\pi\theta_0, \gamma_0 \sin 2\pi\theta_0).$$

Further, there exists $n_0 \in \mathbf{Z}^+$ such that $n_0 + \theta_0 \leq \gamma_0 < n_0 + 1 + \theta_0$. Thus,

$$\begin{aligned} &dist((\eta_0,\xi_0), A) \\ &\qquad = dist(\gamma_0, \{ n_0 + \theta_0 + k/M : 0 \leq k < M \}) \leq \frac{1}{2M}. \end{aligned}$$

Then, as in Example 11, we can construct a discrete subset of A, which is a Fourier frame for $PW_{\bar{B}(0,r)}$.

5. FRAME BOUNDS

We can estimate the frame constants, which are used to implement the results of Section 3 in conjunction with the signal reconstruction formulas (1). These estimates are contained in the following theorems.

Theorem 13 *Let $E \subseteq \mathbf{R}^d$ be a convex, compact set which is symmetric with respect to every coordinate axis and has non-empty interior. Let $\Lambda \subseteq \hat{\mathbf{R}}^d$ be uniformly discrete, and assume*

$$\bigcup_{\lambda\in\Lambda} \tau_\lambda E^* = \hat{\mathbf{R}}^d.$$

Then

$$A\|\phi\|^2 \leq \sum_{\lambda\in\Lambda} |\phi(\lambda)|^2,$$

for all $\phi \in PW_{rE}$, where $r < \frac{\ln 2}{2d\pi}$ and $A = \frac{(2-e^{2d\pi r})^2}{m(E^*)}$.

Theorem 14 *Let $E \subseteq \mathbf{R}^d$ be a convex, compact set which is symmetric with respect to the origin and has non-empty interior, and let Λ be a uniformly discrete set. Then there is $r > 0$ such that*

$$\sum_{\lambda\in\Lambda} |\phi(\lambda)|^2 \leq B\|\phi\|^2,$$

for all $\phi \in PW_E$, where $B = \left(\frac{2}{\pi r}\right)^d e^{4\pi\|(r,\cdots,r)\|_{E^*}}$. (We can take any $r > 0$ for which $\tau_\lambda [-r,r]^d \bigcap \tau_\eta [-r,r]^d = \emptyset$ for any distinct $\lambda, \eta \in \Lambda$.)

6. REFERENCES

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