THE CONDITION NUMBER OF CERTAIN MATRICES AND APPLICATIONS

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ABSTRACT

This paper addresses the problem of estimating the eigenvalues and condition numbers of matrices of the form $R = r(t_i - t_j)$. We begin by mentioning some of the problems in which such matrices occur, and to which the results obtained in this paper may be applied. Examples of such problems include (i) approximation by sums of irregular translates (ii) the missing data problem and incomplete sampling series. Then we describe the method for estimating the eigenvalues and the condition number. Some open issues will also be discussed.

1. INTRODUCTION AND MOTIVATION

Matrices of the form $R_{ij} = r(x_i - x_j)$ often occur in applications. They arise naturally in the context of L_2 (mean-square) approximation by weighted sums of the translates of a single prototype function ϕ. They are also related to the problem of finding conditions under which the translates of ϕ become a Riesz basis. But such matrices also occur in the missing data problem, in the context of sampling expansions, both in the continuous-time and discrete-time settings. We will briefly mention some of these connections.

Let $w \in \mathbb{C}^n$, and write $w \triangleq \{w_i\}_{1 \le i \le n}$. Consider $\xi(w) : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$\xi(w) \triangleq \|f(t) - \sum_{k=1}^{n} w_k \, \phi_k(t)\|^2,$$

where $f \in L_2$, $\{\phi_i\}_{1 \le i \le n}$ are independent L_2 functions, and $\|\cdot\|$ is the L_2 norm[1]. Assume that ξ is to be minimized by adjusting the weights $\{w_i\}_{1 \le i \le n}$. It is widely known [2] that the solution to this L_2 approximation problem can be obtained by solving the so-called normal equations

$$Rw = v,$$

where the matrix R is defined by

$$R \triangleq [\, R_{ij} \,]_{i,j=1}^{n} \triangleq [\, \langle \phi_j, \phi_i \rangle \,]_{i,j=1}^{n}$$

and the elements of the vector v are

$$v_i = \langle f, \phi_i \rangle \, .$$

The approximation by the translates of a fixed prototype function ϕ is a special case of this problem, in which

$$\phi_i(t) \triangleq \phi(t - t_i).$$

In this case, the elements of the matrix R are given by

$$R_{ij} = \langle \phi(\cdot - t_j), \phi(\cdot - t_i) \rangle = \langle \phi(\cdot), \phi(\cdot - t_i + t_j) \rangle \, .$$

Defining the autocorrelation function

$$r(t) = \langle \phi(\cdot), \phi(\cdot - t) \rangle$$

one can write

$$R_{ij} = r(t_i - t_j).$$

Assuming the linear independence of the ϕ_i, the matrix R is positive definite and the interpolation problem

$$f(t_i) = \sum_{j=1}^{n} w_j \, r(t_i - t_j), \quad (1 \le i \le n)$$

is solvable (there exists a $w \in \mathbb{C}^n$ such that the n equations are simultaneously satisfied). The numerical stability of the solution depends on the eigenvalues of R, that is, on the extrema of the quadratic form $x^H R x$,

$$\sum_{i,j} w_i^* w_j \, r(t_i - t_j)$$

subject to $\|w\| = 1$. Let A and B, respectively, denote lower and upper bounds for the eigenvalues of R. Then,

$$A\|w\|^2 \le \sum_{i,j} w_i^* w_j \, r(t_i - t_j) \le B\|w\|^2.$$

[1] The same symbol also denotes the Euclidean norms in $\mathbb{R}^n$ or $\mathbb{C}^n$, but the meaning will be clear from the context.

Using $R_{ij} = r(t_i - t_j) = \langle \phi(\cdot - t_j), \phi(\cdot - t_i) \rangle$ this becomes

$$A\|w\|^2 \le \left\langle \sum_j w_j\, \phi(\cdot - t_j), \sum_i w_i\, \phi(\cdot - t_i) \right\rangle \le B\|w\|^2.$$

This shows how finding bounds A and B for the eigenvalues of R relates to the problem of finding conditions under which $\phi_i(t) \triangleq \phi(t - t_i)$ is a Riesz basis (or an exact frame), with Riesz bounds A and B (see [7, p. 32]). The two problems are equivalent in the subspace generated by the weighted linear combinations of the $\phi_i(t)$,

$$\sum_i w_i\, \phi(t - t_i).$$

The matrices of the form $R_{ij} = r(t_i - t_j)$ are also interesting due to the another reason. Assume that one has a convergent expansion of the form

$$f(t) = \sum_j w_j\, r(t - t_j).$$

We say that this is a sampling expansion if $w_j = f(t_j)$. The simplest example is the sampling series

$$f(t) = \sum_j f(j)\operatorname{sinc}(t - j),$$

the sinc function being defined by

$$\operatorname{sinc} x \triangleq \begin{cases} 1, & x = 0, \\ \frac{\sin \pi x}{\pi x}, & x \neq 0. \end{cases}$$

These sampling expansions have a rich mathematical structure [8, 6] and are also important in digital signal processing, where they have many applications. It often happens that some of the sampled values are unavailable. Let the unknown samples be $\{t_i\}_{i \in U}$, U being some set of distinct integers. Under certain conditions, the unknown samples $f(t_i)$ can be evaluated by solving the equations

$$f(t_i) = \sum_{j \in U} f(t_j)\, r(t_i - t_j) + \sum_{j \notin U} f(t_j)\, r(t_i - t_j), \quad (i \in U),$$

in which the matrix $R_{ij} = r(t_i - t_j)$ again appears. The equations can be solved iff $I - R$ is nonsingular. In the sinc case, when the cardinal of U is finite, R is positive definite and all its eigenvalues belong to the interval $(0, 1)$. This remains true under slightly more general conditions [3]. A study of the stability of the problem can be found in [4, 5] (assuming that the points t_i lie in a regular grid, an assumption that will not be made here).

An additional remark: the sinc function is a reproducing kernel for a certain Paley-Wiener space, and so

$$f(t) = \langle f(\cdot), \operatorname{sinc}(t - \cdot) \rangle ,$$

for any f belonging to the space. Also,

$$\operatorname{sinc} t = \langle \operatorname{sinc}(\cdot), \operatorname{sinc}(t - \cdot) \rangle ,$$

that is, the sinc function is also its own autocorrelation. Thus, conditions on the distribution of the points t_i that ensure that $R_{ij} = r(t_i - t_j)$ has spectral radius $\rho(R) < 1$ (uniformly in n) will also be useful to show that $\operatorname{sinc}(\cdot - t_i)$ is a Riesz basis (or an exact frame) for a certain space. The eigenvalue bounds might then be used to find the Riesz or frame bounds.

2. RESULTS

For a given r, $0 < r < 1$, consider the function

$$s(x) = \begin{cases} 1, & |x| \le r/2, \\ 0, & |x| > r/2. \end{cases}$$

and its Fourier transform $\hat{s}(\xi)$

$$\hat{s}(\xi) = \int_{-r/2}^{+r/2} e^{-\mathrm{i}2\pi\xi x}\, dx = \frac{\sin(\pi r \xi)}{\pi \xi}. \qquad (1)$$

Introducing the sinc function, this becomes

$$\hat{s}(\xi) = r \operatorname{sinc}(r\xi).$$

Given a set of n integers $\{i_k\}_{k=1}^n$, consider the matrix

$$S = [\, S_{ab} \,]_{a,b=1}^n \triangleq [\, \hat{s}(i_a - i_b) \,]_{a,b=1}^n \qquad (2)$$

In addition to the function s and the matrix S, we will need two functions s^+ and s^-, satisfying the inequality

$$0 \le s^-(x) \le s(x) \le s^+(x). \qquad (3)$$

These functions generate the matrices

$$S^+ = [\, S^+_{ab} \,]_{a,b=1}^n \triangleq [\, \hat{s}^+(i_a - i_b) \,]_{a,b=1}^n$$

$$S^- = [\, S^-_{ab} \,]_{a,b=1}^n \triangleq [\, \hat{s}^-(i_a - i_b) \,]_{a,b=1}^n$$

The quadratic form associated with S can be written, using (1),

$$v^H S v = \int_{-r/2}^{+r/2} \left| \sum_{k=1}^n v_k e^{-\mathrm{i}2\pi i_k x} \right|^2 dx = \int_{\mathbb{R}} s(x) |P(x)|^2\, dx, \qquad (4)$$

where

$$P(x) \triangleq \sum_{k=1}^n v_k e^{-\mathrm{i}2\pi i_k x}.$$

An equation similar to (4) holds true for S^+ and S^-.

The eigenvalues of S are the values assumed by the associated quadratic form at its stationary points, under the constraint $\|v\| = 1$. In particular, the extreme eigenvalues of S are given by the maximum and minimum value of the quadratic form as v runs over the unit ball in $\mathbb{C}^n$. Similarly for S^+ and S^-.

This and (3) show that the three matrices S, S^+ and S^- are nonnegative definite. They satisfy the inequalities

$$0 \le S^- \le S \le S^+, \tag{5}$$

whereas their eigenvalues satisfy

$$\begin{array}{ccccc} \lambda_{\max}(S^-) & \le & \lambda_{\max}(S) & \le & \lambda_{\max}(S^+), \\ \lambda_{\min}(S^-) & \le & \lambda_{\min}(S) & \le & \lambda_{\min}(S^+). \end{array} \tag{6}$$

We will now consider the problem of bounding the eigenvalues and condition number of the matrix S defined by (2). The bounds will be obtained assuming that the $\{i_k\}_{k=1}^n$ satisfy

$$|i_a - i_b| \ge m|a-b|$$

for some $m > 1$. This condition is much weaker than the one assumed in [4], and the bounds that will be obtained generalize those presented in that paper.

Noting that $S_{ii} = \hat{s}(0) = r$, one sees that the Geršgorin discs associated with the matrix S are the sets

$$D_i \triangleq \{z \in \mathbb{C} : |z - r| \le R_i(S)\},$$

where

$$R_i(S) \triangleq \sum_{\substack{j=0 \\ j \ne i}}^{n-1} |S_{ij}|.$$

These discs are not very useful because S is not in general diagonally dominant. This happens because the function $s(x)$ is discontinuous. In fact, $\hat{s}(\xi)$ is $O(1/\xi)$, and the radii $R_i(S)$ will diverge as the size n of the matrix S increases, unless the $\{i_k\}_{k=1}^n$ are very sparse.

To avoid this problem we consider the discs for S^+ and S^-. We start by estimating $R_i(S^+)$ and $R_i(S^-)$ for certain S^+ and S^-. The results will be used to bound the eigenvalues of S itself (see (6)). The condition number of $I - S$ is given by

$$\kappa(I-S) \triangleq \frac{\lambda_{\max}(I-S)}{\lambda_{\min}(I-S)} = \frac{1 - \lambda_{\min}(S)}{1 - \lambda_{\max}(S)}.$$

The bounds

$$\lambda_{\max}(S) \le \lambda_{\max}(S^+) \le \hat{s}^+(0) + \max_i R_i(S^+), \tag{7}$$

$$\lambda_{\min}(S) \ge \lambda_{\min}(S^-) \ge \hat{s}^-(0) - \max_i R_i(S^-), \tag{8}$$

lead to

$$\kappa(I-S) \le \frac{1 - \lambda_{\min}(S^-)}{1 - \lambda_{\max}(S^+)} \le \frac{1 - \hat{s}^-(0) + \max_i R_i(S^-)}{1 - \hat{s}^+(0) - \max_i R_i(S^+)}. \tag{9}$$

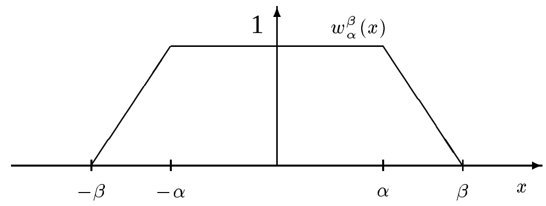

Figure 1: The function $w_\alpha^\beta(x)$, depending on the parameters α and β, $\beta > \alpha$.

To begin we need to select two candidate functions s^+ and s^-, and the corresponding matrices S^+ and S^-. The simplest continuous function upon which s^+ and s^- can be based is probably the trapezoidal function w_α^β depicted in the figure 1. Its Fourier transform $\hat{w}_\alpha^\beta(\xi)$

$$\hat{w}_\alpha^\beta(\xi) \triangleq \int_{-\infty}^{+\infty} w_\alpha^\beta(x) e^{-\mathrm{i}2\pi\xi x}\, dx$$

is given by

$$\hat{w}_\alpha^\beta(\xi) = \frac{\sin[\pi(\alpha+\beta)\xi]\ \sin[\pi(\beta-\alpha)\xi]}{\pi^2(\beta-\alpha)\xi^2},$$

that is,

$$\hat{w}_\alpha^\beta(\xi) = (\alpha+\beta)\,\mathrm{sinc}[(\alpha+\beta)\xi]\,\mathrm{sinc}[(\beta-\alpha)\xi].$$

It follows that

$$\begin{aligned} \sum_{a \ne b} |\hat{w}_\alpha^\beta(i_a - i_b)| &\le \frac{1}{\pi^2(\beta-\alpha)} \sum_{a\ne b} \frac{1}{(i_a - i_b)^2} \\ &\le \frac{1}{\pi^2(\beta-\alpha)} \sum_{a \ne b} \frac{1}{m^2(a-b)^2} \\ &\le \frac{2}{\pi^2 m^2(\beta-\alpha)} \sum_{k=1}^{\infty} \frac{1}{k^2} \\ &\le \frac{1}{3m^2(\beta-\alpha)}, \end{aligned} \tag{10}$$

using Euler's classical result $\sum_1^\infty 1/k^2 = \pi^2/6$. Note that

$$\hat{w}_\alpha^\beta(0) = \alpha + \beta. \tag{11}$$

We will take $\alpha = r/2$ and $\beta = r/2 + \delta$, where δ is any positive real, to specify $s^+(x)$. This and (11) means that

$$\hat{s}^+(0) = r + \delta, \tag{12}$$

and using (10) one sees that

$$R_i(S^+) \le \frac{1}{3m^2\delta}. \tag{13}$$

Inserting (12) and (13) in (7) leads to

$$\lambda_{\max}(S) \le r + \delta + \frac{1}{3m^2\delta}.$$

The minimum value of the bound is obtained for $\delta = 1/(\sqrt{3}m)$, and is

$$\lambda_{\max}(S) \leq r + \frac{2}{\sqrt{3}m}. \qquad (14)$$

Having dealt with s^+ and the upper bound, we now turn to s^-. We will take $\alpha = r/2 - \delta$ and $\beta = r/2$, where δ is any positive real, to specify $s^-(x)$. This and (11) means that

$$\hat{s}^-(0) = r - \delta, \qquad (15)$$

and using (10) one sees that, as in the previous case,

$$R_i(S^-) \leq \frac{1}{3m^2\delta}. \qquad (16)$$

Inserting (15) and (16) in (8) leads to

$$\lambda_{\min}(S) \geq r - \delta - \frac{1}{3m^2\delta}.$$

The maximum value of the bound is obtained for $\delta = 1/(\sqrt{3}m)$, and is

$$\lambda_{\min}(S) \geq r - \frac{2}{\sqrt{3}m}. \qquad (17)$$

Combining (14) and (17), or using (9) directly,

$$\kappa(I - S) \leq \frac{1 - r + \frac{2}{\sqrt{3}m}}{1 - r - \frac{2}{\sqrt{3}m}},$$

which is the required bound for the condition number of $I - S$.

Several remarks can be made. First, the bounds for the eigenvalues and the condition number apply for matrices other than S (indeed, to the matrices generated by any other band-limited function f the Fourier transform of which can be bounded by trapezoidal functions). Second, many functions other than the trapezoidal functions used will lead to valid bounds. Using truncated cosine functions, for example, leads to the bound

$$\kappa(I - S) \leq \frac{1 - \frac{1}{m}\cot\frac{\pi}{2rm}}{1 - \frac{4\alpha\beta}{\pi} + \frac{1}{m}\cot\frac{\pi}{2\beta m}}.$$

The problem of selecting the function that leads to the best possible bounds is open.

3. REFERENCES

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