BAYESIAN IMAGE RESTORATION USING A WAVELET-BASED SUBBAND DECOMPOSITION

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ABSTRACT

In this paper the subband decomposition of a single channel image restoration problem is examined. The decomposition is carried out in the image model (prior model) in order to take into account the frequency activity of each band of the original image. The hyperparameters associated with each band together with the original image are rigorously estimated within the Bayesian framework. Finally, the proposed method is tested and compared with other methods on real images.

1. INTRODUCTION

A standard formulation of the image degradation model is given in lexicographic form by [1]

$$\mathbf{g} = \mathbf{D}\mathbf{f} + \mathbf{w}, \tag{1}$$

where the $p \times 1$ vectors $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{w}$ represent respectively the original image, the available noisy and blurred image and the noise with independent elements of variance $\sigma^2_{\mathbf{w}} = \beta^{-1}$, and $\mathbf{D}$ represents the known blurring matrix. The images are assumed to be of size $n \times n$, with $p = n \times n$. The *restoration problem* calls for finding an estimate of $\mathbf{f}$ given $\mathbf{g}$, $\mathbf{D}$ and knowledge about $\mathbf{w}$ and possibly $\mathbf{f}$ (see Chapter 1 in [7]).

Smoothness constraints on the original image can be incorporated under the form of

$$p(\mathbf{f}|\alpha) \propto \alpha^{p/2} \exp\{-\frac{1}{2}\alpha \parallel \mathbf{C}\mathbf{f} \parallel^2\}, \tag{2}$$

where $\mathbf{C}$ is the Laplacian operator.

Then, following the Bayesian paradigm it is customary to select as the restoration of $\mathbf{f}$, the image $\mathbf{f}_{(\alpha,\beta)}$ defined by

$$\begin{aligned} \mathbf{f}_{(\alpha,\beta)} &= \arg\{ \min_{\mathbf{f}} [\, \alpha \parallel \mathbf{C}\mathbf{f} \parallel^2 + \beta \parallel \mathbf{g} - \mathbf{D}\mathbf{f} \parallel^2] \} \\ &= \arg\{ \max_{\mathbf{f}} p(\mathbf{f}|\alpha) p(\mathbf{g}|\mathbf{f},\beta) \}, \end{aligned} \tag{3}$$

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where from Eq. 1 we have

$$p(\mathbf{g}|\mathbf{f},\beta) \propto \beta^{p/2} \exp\{-\frac{1}{2}\beta \parallel \mathbf{g} - \mathbf{D}\mathbf{f} \parallel^2\}\,. \tag{4}$$

An important problem arises when α and/or β are unknown. Much interest has centered on the question of how these parameters should be estimated (see [6], [9]). It is widely accepted that the hyperparameter in the image model (α) should be adapted to the local image characteristics.

The application of multichannel techniques to single channel restoration problems using a subband decomposition was proposed in [2] and [3] using the framework developed in [8].

In this paper we examine the subband decomposition of the quadratic image model given in Eq. 2. Since by performing a subband decomposition we are extracting different frequency regions (channels) of an image, the process of associating a different image hyperparameter to each subband of the image model becomes equivalent to assigning different hyperparameters to different frequency bands in the image. These hyperparameters will reflect then the activity of that band in the original image. We show how the estimation of these parameters can be carried out within the Bayesian image restoration paradigm.

The rest of the paper is organized as follows. In section 2 the image and noise models are defined in order to apply the Bayesian paradigm. For those image and noise models, the estimation of the hyperparameters and the original image is performed in section 3. Finally, in section 4 experimental results are shown and section 5 concludes the paper.

2. IMAGE AND NOISE MODELS

A simple way to incorporate the smoothness of the object luminosity is to model the distribution of $\mathbf{f}$ by Eq. 2. It is

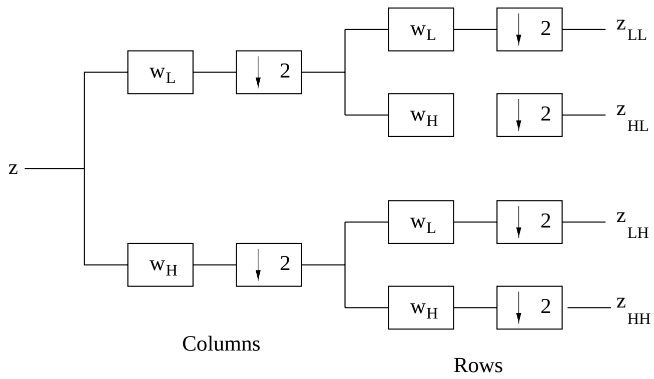

Figure 1: Four-channel 2-D decomposition.

important to note that this model is a simultaneous autorregression (SAR) ([10]) and is characterized by

$$(\mathbf{Cf})_i = \epsilon_i, \quad (5)$$

where the ϵ_is are independent $\mathcal{N}(0, \alpha^{-1})$.

A careful examination of Eq. 5 shows that this expression is not true for real images. The spectrum of $\mathbf{Cf}$ is not normally flat and the energy in each frequency is not the same (equal to $1/\alpha$). Obviously the image model is just a simple approximation.

Let us now consider $\mathbf{z} = \mathbf{Cf}$ and perform a multichannel decomposition on it. Let $\mathbf{w}_l$ and $\mathbf{w}_h$ be $1-D$ quadrature mirror filters (QMF) based on the orthonormal wavelet bases with compact support ([4]), so that one set of coefficients may be used to defined the other ([11]). Then, the subband decomposition of $\mathbf{z}$ can be calculated as described in Fig. 1.

We note that

$$\mathbf{I} = \mathbf{W}_{ll}^t\mathbf{W}_{ll} + \mathbf{W}_{hl}^t\mathbf{W}_{hl} + \mathbf{W}_{lh}^t\mathbf{W}_{lh} + \mathbf{W}_{hh}^t\mathbf{W}_{hh}, \quad (6)$$

where $\mathbf{W}_{uv}$ with $u, v \in l, h$ are the $[(n/2)\times(n/2)] \times [n \times n]$ matrices used to obtain the bands $\mathbf{z}_{uv}$ (see Fig. 1) and ^{t} denotes transpose. It is important to observe that now $\mathbf{W}_{uv}^t\mathbf{W}_{uv}\mathbf{z}$ contains information on some part of the spectrum of $\mathbf{z}$.

Let us consider the quadratic form defining the image model; we have

$$\begin{aligned} \alpha \parallel \mathbf{Cf} \parallel^2 &= \alpha\mathbf{f}^t\mathbf{C}^t\mathbf{Cf} \\ &= \mathbf{f}^t\mathbf{C}^t(\alpha\mathbf{W}_{ll}^t\mathbf{W}_{ll} + \alpha\mathbf{W}_{hl}^t\mathbf{W}_{hl} + \\ &\quad \alpha\mathbf{W}_{lh}^t\mathbf{W}_{lh} + \alpha\mathbf{W}_{hh}^t\mathbf{W}_{hh})\mathbf{Cf}. \quad (7) \end{aligned}$$

Now, in order to adapt the image model, and therefore have a hyperparameter for each of the decomposed channels, we define the following image model

$$p(\mathbf{f}|\underline{\alpha}) \propto \frac{1}{Z_{prior}(\underline{\alpha})}\exp\{-\frac{1}{2}\sum_{u,v\in\{l,h\}}\alpha_{uv} \parallel \mathbf{W}_{uv}\mathbf{Cf} \parallel^2\}, \quad (8)$$

where $\underline{\alpha}$ denotes the vector $(\alpha_{ll}, \alpha_{hl}, \alpha_{lh}, \alpha_{hh})$ and

$$Z_{prior}(\underline{\alpha}) = |\mathbf{P}(\underline{\alpha})|^{-1/2}, \quad (9)$$

where $\mathbf{P}(\underline{\alpha}) = \sum_{u,v\in\{l,h\}}\alpha_{u,v}\mathbf{C}^t\mathbf{W}_{uv}^t\mathbf{W}_{uv}\mathbf{C}$.

Note that the model we have just proposed can be extended to a 4^i-channel decomposition. However, for notational simplicity, we will only use a 4-channel decomposition. We also note that the image model we are proposing allows the use of the same hyperparameters for several subbands.

Let us now examine how to estimate the unknown parameters and perform the restoration in the coming section.

3. BAYESIAN ANALYSIS

The steps we follow in this paper to estimate the hyperparameters and the original image are

Step I: Estimation of the hyperparameters

$\hat{\underline{\alpha}} = (\hat{\alpha}_{ll}, \hat{\alpha}_{hl}, \hat{\alpha}_{lh}, \hat{\alpha}_{ll})$ and $\hat{\beta}$ are first selected as

$$\hat{\underline{\alpha}}, \hat{\beta} = \arg\max_{\underline{\alpha},\beta}\mathcal{L}_{\mathbf{g}}(\underline{\alpha}, \beta) = \arg\max_{\underline{\alpha},\beta}\log p(\mathbf{g}|\underline{\alpha}, \beta), \quad (10)$$

where $p(\mathbf{g}|\underline{\alpha}, \beta) = \int_{\mathbf{f}} p(\mathbf{f}|\underline{\alpha})p(\mathbf{g}|\mathbf{f}, \beta)d\mathbf{f}$.

Step II: Estimation of the original image

Once the hyperparamenters have been estimated, the estimation of the original image, $\mathbf{f}_{(\hat{\underline{\alpha}},\hat{\beta})}$, is selected as the image satisfying

$$\mathbf{f}_{(\hat{\underline{\alpha}},\hat{\beta})} = \arg\min_{\mathbf{f}}\sum_{uv\in\{lh\}}\hat{\alpha}_{uv} \parallel \mathbf{W}_{uv}\mathbf{Cf} \parallel^2 + \hat{\beta} \parallel \mathbf{g}-\mathbf{Df} \parallel^2 \quad (11)$$

Note that we are obtaining the maximum likelihood estimates of the hyperparameters and the *maximum a posteriori* (MAP) estimate of $\mathbf{f}$. Furthermore, although steps I and II are separated, the iterative scheme proposed next performs both estimations simultaneously.

The estimation process we are using could be performed within the so called hierarchical Bayesian approach (see [9]) by including hyperpriors on the unknown hypervector $\hat{\underline{\alpha}}$ and hyperparameter $\hat{\beta}$. However, the possibility of incorporating additional knowledge on them by means of gamma or other distributions will not be discussed here (see [9]).

Differentiating $-2\mathcal{L}_{\mathbf{g}}(\underline{\alpha}, \beta)$ with respect to $\underline{\alpha}$ and β so as to find the conditions which are satisfied at the maxima we have

$$\begin{aligned} &\parallel \mathbf{W}_{uv}\mathbf{Cf}_{(\underline{\alpha},\beta)} \parallel^2 + trace[\mathbf{Q}(\underline{\alpha}, \beta)^{-1}\mathbf{C}^t\mathbf{W}_{uv}^t\mathbf{W}_{uv}\mathbf{C}] = \\ &\quad trace[\mathbf{P}(\underline{\alpha})^{-1}\mathbf{C}^t\mathbf{W}_{uv}^t\mathbf{W}_{uv}\mathbf{C}] \quad \text{for } u, v \in \{l, h\} \quad (12) \\ &\parallel \mathbf{g}-\mathbf{Df}_{(\underline{\alpha},\beta)} \parallel^2 + trace[\mathbf{Q}(\underline{\alpha}, \beta)^{-1}\mathbf{D}^t\mathbf{D}] = p/\beta, \quad (13) \end{aligned}$$

where $\mathbf{Q}(\underline{\alpha}, \beta) = \sum_{u,v \in \{l,h\}} \alpha_{u,v} \mathbf{C}^t \mathbf{W}_{u,v}^t \mathbf{W}_{u,v} \mathbf{C} + \mathbf{D}^t \mathbf{D}$.

Let us examine the use of the EM-algorithm [5] with $\mathcal{X}^t = (\mathbf{f}^t, \mathbf{g}^t)$ and $\mathcal{Y} = \mathbf{g} = [\mathbf{0} \quad \mathbf{I}]^{\mathbf{t}} \mathcal{X}$ to iteratively increase $\mathcal{L}_{\mathbf{g}}(\underline{\alpha}, \beta)$. The application of the EM-algorithm to our problem produces Eqs. 12 and 13 where the old values of the hyperparameters are used on the left hand side of these equations to obtain the new ones on their right hand side. Unfortunately, these equations are highly nonlinear.

Let us, however, consider first the iterative EM equations corresponding to using one hyperparameter for the image model (α) and one for the noise (β) (see [9]). We have

$$\left[\frac{1}{\alpha}\right]_{new} = \left[\frac{1}{p}\left\{\| \mathbf{C}\mathbf{f}_{(\alpha,\beta)} \|^2 + trace[(\alpha \mathbf{C}^t\mathbf{C} + \beta \mathbf{D}^t \mathbf{D})^{-1}\mathbf{C}^t\mathbf{C}]\right\}\right]_{old} \quad (14)$$

$$\left[\frac{1}{\beta}\right]_{new} = \left[\frac{1}{p}\left\{\| \mathbf{g} - \mathbf{D}\mathbf{f}_{(\alpha,\beta)} \|^2 + trace[(\alpha \mathbf{C}^t\mathbf{C} + \beta \mathbf{D}^t \mathbf{D})^{-1}\mathbf{D}^t\mathbf{D}]\right\}\right]_{old}, \quad (15)$$

where $\mathbf{f}_{(\alpha,\beta)}$ has been defined in Eq. 3 and $[\ ]_{new}$ and $[\ ]_{old}$ represent the evaluation of the expressions for the new and old values of α and β, respectively. We notice that these equations correspond to the application of a gradient descendent method on $1/\alpha$ and $1/\beta$.

Let us adapt this method to the multichannel problem. Multiplying and dividing the right hand side of Eq. 12 by α_{uv} we have

$$\frac{1}{p(\alpha_{uv})}\left[\| \mathbf{W}_{uv}\mathbf{C}\mathbf{f}_{(\underline{\alpha},\beta)} \|^2 + trace[\mathbf{Q}(\underline{\alpha},\beta)^{-1}\mathbf{C}^t W_{uv}^t \mathbf{W}_{uv}\mathbf{C}]\right] = 1/\alpha_{uv}, \quad (16)$$

where $p(\alpha_{uv}) = \alpha_{uv} trace[\mathbf{P}(\underline{\alpha})^{-1}\mathbf{C}^t\mathbf{W}_{uv}^t\mathbf{W}_{uv}\mathbf{C}]$ (we have removed the dependency on $\underline{\alpha}$ of $p(\alpha_{uv})$ to simplify the notation). Notice that $p(\alpha_{uv}) = p$ if we have only one image parameter and that $\sum_{u,v \in \{l,h\}} p(\alpha_{uv}) = p$.

Then, we can use the following equations to estimate the hyperparameters, where the old values are used in the right hand side of the equations to obtain the new ones on the left hand side

$$\left[\frac{1}{\alpha_{uv}}\right]_{new} = \left[\frac{1}{p(\alpha_{uv})}\left(\| \mathbf{W}_{uv}\mathbf{C}\mathbf{f}_{(\underline{\alpha},\beta)} \|^2 + trace[\mathbf{Q}(\underline{\alpha},\beta)^{-1}\mathbf{C}^t\mathbf{W}_{uv}^t\mathbf{W}_{uv}\mathbf{C}]\right)\right]_{old} \quad \text{for } u,v \in \{l,h\} \quad (17)$$

$$\left[\frac{1}{\beta}\right]_{new} = \left[\frac{1}{p}\left\{\| \mathbf{g} - \mathbf{D}\mathbf{f}_{(\underline{\alpha},\beta)} \|^2 + trace[\mathbf{Q}(\underline{\alpha},\beta)^{-1}\mathbf{D}^t\mathbf{D}]\right\}\right]_{old} \quad (18)$$

This method is again a gradient descendent one. We have used it in our experiments and have not observed any convergence problem, however, it would always be possible to use smaller steps as to guarantee convergence.

dB	Method	Iterations	$ISNR$	β^{-1}
10	MLE	40	7.6038	184.98
	1 param.	30	7.2863	214.03
	2 params.	50	7.2844	214.09
	4 params.	60	7.2241	213.95
20	MLE	35	7.9889	49.84
	1 param.	35	8.8162	63.64
	2 params.	50	8.8086	63.68
	4 params.	70	8.2787	64.73
30	MLE	30	6.2049	3.91
	1 param.	35	8.8181	6.37
	2 params.	41	8.8006	6.42
	4 params.	90	9.1402	6.57

Table 1: Iterations required, ISNR, and noise variance estimations for the "Cameraman" image and different SNRs.

4. EXPERIMENTAL RESULTS

In order to show the behavior of the proposed algorithm, we have used the original 256×256 "Cameraman" image, blurred by a motion blur over 9 pixels. It was also degraded by additive Gaussian noise to achieve 10, 20 and 30dB SNR (noise variances of $\beta^{-1} = 216.1$, $\beta^{-1} = 64$, and $\beta^{-1} = 6.25$, respectively). For a comparison, we have also applied the maximum likelihood restoration method to these degraded images.

For the purpose of objectively testing the performance of the image restoration algorithms, the Improvement in Signal to Noise Ratio (ISNR) will be used. This metric is given by

$$ISNR = 10\log_{10}\left\{\frac{\sum_{m,n}[\mathbf{f}(m,n) - \mathbf{g}(m,n)]^2}{\sum_{m,n}\left[\mathbf{f}(m,n) - \mathbf{f}_{(\underline{\hat{\alpha}},\hat{\beta})}(m,n)\right]^2}\right\}$$

The values of ISNR, the required number of iterations needed to achieve convergence in parameter estimation, and the corresponding values of the estimated noise variance are shown in Table 1. We have included the results obtained by maximum likelihood and the proposed algorithm using only one parameter for all the bands of the image; using two parameters, one for the ll band, and a different one for the lh, hl, and hh bands. The last row of every sub-table contains the result obtained using a different parameter for each subband. The set of coefficients used is DAUB4.

From this table we can see that the proposed method results in better estimates of the noise variance, very close to the real value, giving less noisy images than the maximum

(a)

(b)

(c)

(d)

Figure 2: (a) Original "Cameraman" image. (b) Noisy-blurred image for 9-point motion blur at 30dB. (c) Maximum likelihood restoration. (d) Restoration obtained with the proposed method.

likelihood method. We can see that the ISNR is in general better for the proposed method. The 10dB case is the only one where maximum likelihood gives better results in terms of ISNR, but the images obtained using the proposed algorithm appear to be better from a subjective point of view (visual inspection).

Fig. 2 shows the original "Cameraman" image, the degraded image at 30 dB, the restoration obtained by maximum likelihood and the restoration obtained with the proposed method using four prior-model parameters, a different one for each band. We can see that the solution proposed gives smoother solutions but the noise is much better removed.

5. CONCLUSIONS

In this paper we have proposed the decomposition of the single channel image restoration problem in order to take into account the frequency activity in each subband of the decomposed image. The Bayesian framework has been used to estimate both the parameters and the restored image.

The results obtained using the proposed method have been compared to those obtained by the maximum likelihood restoration method. The proposed method results in better estimates of the parameters involved in the problem, giving less noisy results. We have also used objective metrics to measure the quality of the resulting restorations. In general better solutions are obtained with the proposed approach, than with the maximum likelihood method.

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