

# FIXED POINT ALGORITHM FOR BIT RATE OPTIMAL EQUALIZATION IN MULTICARRIER SYSTEMS

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## ABSTRACT

In this paper we present a fixed point algorithm for bit rate optimal time domain equalization in multicarrier systems. The algorithm outperforms standard MMSE equalizers in capacity measure. Moreover, it adaptively estimates the optimal signal delay which was an exhaustive search problem in MMSE equalizers. We briefly discuss the convergence properties and demonstrate the performance of the algorithm on a twisted pair copper wire loop with additive non-white noise.

## 1. INTRODUCTION

Multicarrier modulation (MCM) systems have become the technology of choice for high-speed data transmission over spectrally shaped channels. MCM transceivers divide the available frequency spectrum into a large number of parallel, independent and approximately flat subchannels, thus directly implementing the maximum likelihood (ML) detectors on the receiver side. The preferred form of multicarrier modulation is the Discrete Multitone Transform (DMT), which uses the computationally effective Fast Fourier Transform to create independent channels. Combined with the optimization of bandwidth and bit distribution across usable subchannels, this modulation scheme demonstrated the results comparable to the optimum eigendecomposition based MCM transceiver of [9, 3, 1].

The fundamental implementational difficulty of the multicarrier transmission is the intersymbol interference (ISI) which renders the subchannels mutually dependent. Intersymbol interference is usually combated by addition of the cyclic prefix whose length is equal to the channel memory, so that the output sequence seems periodic to the channel. This quasi-periodicity makes the channel description matrix circulant. In case of additive white noise, the FFT basis vectors become the eigenvectors of the channel description matrix and its eigenvalues become the DFT coefficients of its first column.

The addition of the cyclic prefix, although very efficient in decoupling the subchannels, does decrease the transmission rate, especially for the channels with long finite impulse response (FIR) and small number of DMT carriers. The equalization in MCM systems is therefore divided into two parts: time domain equalization (TEQ) and frequency domain equalization (FEQ). TEQ focuses on the task of optimally (in a certain sense) shortening of the impulse re-

sponse of the channel to the length of desired cyclic prefix, while FEQ equalizes the frequency response of the channel. After the TEQ the cyclic prefix is discarded together with the intersymbol interference contained in it.

Various approaches to the time domain equalization are possible. Due to ease of their implementation and existence of adaptive algorithms, the minimum mean square error (MMSE) algorithms are most widely used. MMSE algorithms minimize the error between the target impulse response (TIR) and the equalized channel response. The length of the target impulse response is selected to be same as the desired length of the cyclic prefix. The minimization of the mean square error therefore results in minimization of the trailing ISI in the residual energy sense. The solution of the MMSE problem for DMT transceivers exists in the closed form [9, 3]. However, this structure does not guarantee the optimality of the channel capacity, which is the approach we discuss in this paper. The channel capacity optimization algorithms have been discussed by Al-Dhahir and Cioffi [3, 1] in the form of nonlinear optimization problem. Although it has been shown that this approach has a definite performance advantage to MMSE it has also been demonstrated that it is computationally very intensive and has very uncertain convergence properties.

In this paper we analyze the problems behind the capacity optimization in more detail and reveal why most common optimization techniques fail. However, the main contribution of this paper is introduction of a fixed point technique having following benefits:

1. Guaranteed convergence to capacity optimizing values of FIR TEQ equalizer regardless of the starting point,
2. Adaptive determination of optimal signal delay, and
3. No matrix inversion and autocorrelation matrix estimation is required.

The algorithm however, does have substantial memory requirements.

## 2. PROBLEM ANALYSIS

The block diagram of a DMT transmission system is shown in Figure 1. A  $\bar{N}$ -point FFT is used to divide the channel spectrum into  $\bar{N}$  subchannels. For channels to be completely independent and memoryless,  $\bar{N}$  has to be infinite. In practical applications, on the other hand,  $\bar{N}$  is chosen

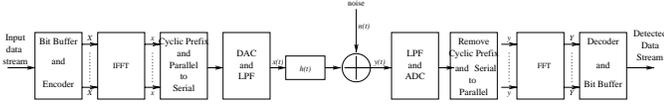


Figure 1: Block diagram of a DMT transmitter.

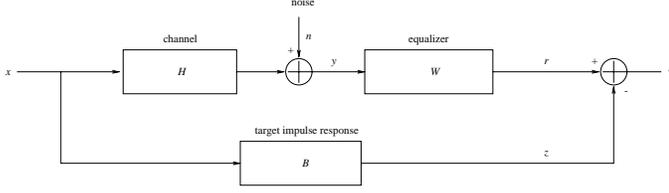


Figure 2: Block diagram of time domain equalization.

to be a power of 2. A common number used is 512. The finite number of subchannels results in the inter-block interference (IBI) which is mitigated by addition of the cyclic prefix (CP) which clears the channel memory thus making the successive block transmissions independent. Thus transformed signal is then low-pass filtered, converted to analog and sent through the channel. On the receiving side the procedure is reversed.

As we indicated before, for highly dispersive channels, the addition of the cyclic prefix whose length is same as the length of the channel response severely decreases the data rate. The solution to this problem as suggested in [9] is to linearly equalize the channel impulse response to a shorter, target impulse response (TIR). Target impulse response is chosen to be closest in the mean square error to the channel-equalizer combination as in Figure 2.

Framework we adopt is one of fractional equalization. The time domain equalizer is taken to be of length  $N_f + 1$  and is denoted by vector  $\mathbf{w} = [w_0, w_1, \dots, w_{N_f}]^T$ . Target impulse response is taken to be of length  $N_b + 1$  and is denoted by  $\mathbf{b} = [b_0, b_1, \dots, b_{N_b}]^T$ .

For any fixed  $\mathbf{b}$ , from the orthogonality principle we have that the filter coefficients of the feed-forward equalizer  $\mathbf{w}$  can be written as:

$$\mathbf{w}^* \mathbf{R}_{yy} = \mathbf{b}^* \mathbf{R}_{xy} \quad (1)$$

where  $\mathbf{R}_{yy}$  and  $\mathbf{R}_{xy}$  are the crosscorrelation matrices. This is the minimum mean square error (MMSE) solution that has been well documented in literature [9, 3, 1].

As it has been noted above, there is no guarantee that this solution will provide us with the maximum channel capacity. In order to investigate the channel capacity problem we compute the number of bits in each of  $\bar{N}$  frequency bins of DMT and sum them to obtain the number of bits transmitted in one DMT symbol:

$$b_{DMT} = \sum_{i=1}^{\bar{N}} b_i = \sum_{i=1}^{\bar{N}} \log_2 \left( 1 + \frac{SNR_i}{\gamma} \right) \quad (2)$$

where  $\gamma$  is the signal to noise ratio gap that characterizes the distance (in dB) between  $SNR_i$  and the signal to noise ratio required to achieve the capacity and it is a function

of the probability of error, required margin and the coding gain. In order to treat all bins equally we set  $\gamma_i = \gamma$ .

The overall number of bits can be expressed as:

$$b_{DMT} = \log_2 \left( 1 + \frac{SNR_{geometric}}{\gamma} \right) \quad (3)$$

where we introduced  $SNR_{geometric} = \left[ \prod_{i=1}^{\bar{N}} \left( 1 + \frac{SNR_i}{\gamma} \right) \right]^{\frac{1}{\bar{N}}}$ . Typically, we assume that factors "1" and "1" can be ignored and simplify the above expression to:

$$SNR_{geometric} \approx \left[ \prod_{i=1}^{\bar{N}} SNR_i \right]^{\frac{1}{\bar{N}}} \quad (4)$$

Since the value of capacity is directly related to the  $SNR_{geometric}$ , from here on we consider only the optimization of this value. For the purpose of analysis we express the  $SNR_{geometric}$  in terms of coefficients of vector  $\mathbf{b}$  as:

$$SNR_{geometric}^n = \prod_{i=1}^{\bar{N}} SNR_i = \prod_{i=1}^{\bar{N}} \frac{\mathbf{b}^* \mathbf{M}_i \mathbf{b}}{\mathbf{b}^* \mathbf{N}_i \mathbf{b}} \quad (5)$$

where

$$\begin{aligned} \mathbf{M}_i &= \mathbf{S}_i^* \Phi_i^{bb}, \\ \mathbf{N}_i &= \mathbf{S}_i^* \mathbf{R}_{xy} \mathbf{R}_{yy}^{-*} \Phi_i^{ww} \mathbf{R}_{yy}^{-1} \mathbf{R}_{xy} + \mathbf{S}_i^* (\Phi_i^{bb} \\ &\quad - \mathbf{H}_i \Phi_i^{bw} \mathbf{R}_{yy}^{-1} \mathbf{R}_{xy} - \mathbf{H}_i^* \mathbf{R}_{yy}^{-*} \mathbf{R}_{xy}^* \Phi_i^{wb} + \\ &\quad |\mathbf{H}_i|^2 \mathbf{R}_{yy}^{-*} \mathbf{R}_{xy} \Phi_i^{bb} \mathbf{R}_{yy}^{-1} \mathbf{R}_{xy}) \end{aligned} \quad (6)$$

and,  $\Phi_i^{bb}$ ,  $\Phi_i^{ww}$ , and  $\Phi_i^{bw}$  are the Fourier matrices such that  $|B[i]|^2 = \mathbf{b}^* \Phi_i^{bb} \mathbf{b}$ ,  $|W[i]|^2 = \mathbf{w}^* \Phi_i^{ww} \mathbf{w}$  and  $W[i]^* B[i] = \mathbf{w}^* \Phi_i^{wb} \mathbf{b}$ .

Since quantity  $\frac{\mathbf{b}^* \mathbf{M}_i \mathbf{b}}{\mathbf{b}^* \mathbf{N}_i \mathbf{b}}$  is recognized as the Rayleigh coefficient, it follows that the vector  $\mathbf{b}$  that maximizes the capacity for bin  $i$  is given as a solution of the generalized eigenvalue problem for the pencil  $[\mathbf{M}_i, \mathbf{N}_i]$ . Unfortunately, this conclusion cannot be generalized for the product of Rayleigh coefficients for each of the frequency bins. It is not clear whether the closed form solution for this problem exists.

Even the numerical optimization of the log-product of Rayleigh coefficients turns out to be cumbersome. First, a simple gradient optimization is computationally intensive since the gradient is given by:

$$\nabla_{\mathbf{b}} SNR_{geometric} = SNR_{geometric} \left[ \frac{\mathbf{b}^* \mathbf{M}_i}{\mathbf{b}^* \mathbf{N}_i \mathbf{b}} - \frac{\mathbf{b}^* \mathbf{N}_i}{\mathbf{b}^* \mathbf{M}_i \mathbf{b}} \right] \quad (7)$$

and, therefore, requires quadratic computational effort.

Moreover, due to the specific structure of matrices  $\mathbf{M}_i$ , the numerator of the  $SNR_{geometric}$  has numerous zeros. For example, when  $N_b = 4$  (i.e. the cyclic prefix is of length 5) and for FFT length of 64, the numerator of  $SNR_{geometric}$  has 101 zero eigenvalues. Moreover, matrices  $\mathbf{N}_i$  can be poorly conditioned and may have the minimum eigenvalue close, or even equal to the small or zero eigenvalue of the numerator. Overall, this structure contributes to the very hard optimization surface with many local minima and maxima and possibly discontinuous gradients.

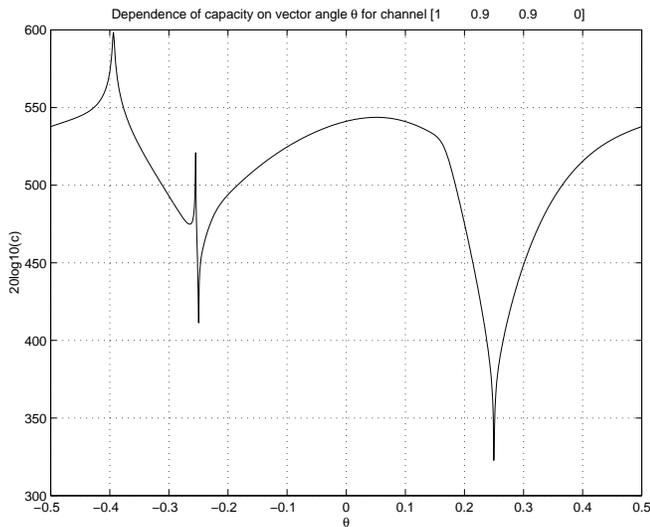


Figure 3: Dependence of capacity on the parameter  $\theta = \arctan b_2/b_1$  for a simple channel.

In order to better illustrate above described behavior we performed a simple exercise that can be easily tracked analytically. We selected the order of filter  $\mathbf{b}$  to be 2, and used a simple channel:  $[1, 0.9, 0.9]$ . We computed the capacity for all values of the angle  $\theta$  that defines the unit norm vector  $\mathbf{b}$  as  $\theta = \tan^{-1}(b_2/b_1)$ . The parameter  $\theta$  describes the whole space of vectors  $\mathbf{b}$ , so the optimization surface is actually a line. The dependence of the capacity on the parameter  $\theta$  is shown in Figure 3.

It is obvious that for a more complex surfaces standard optimization algorithms suffer from locality problems and cannot guarantee the convergence to the optimal capacity.

### 3. FIXED POINT HYBRID EVOLUTIONARY ALGORITHM

#### 3.1. Description of the Algorithm

We propose an alternative way of DMT TEQ filter design well suited for the fixed point DSP processor applications. The algorithm is a hybrid of the Simulated Annealing and Genetic Algorithms and by construction borrows the global convergence from Simulated Annealing and convergence speed from Genetic Algorithm.

We initialize the algorithm with a set (population) of  $N$  column vectors  $\tau = [\mathbf{b}', \mathbf{w}', \Delta]'$  and the control parameter  $T$ .  $\Delta$  here stands for the received data delay required for the best symbol alignment.

For each of the vectors we compute the capacity and rank and sort the vectors with respect to their capacity. At this point it is important to indicate that the capacity computation is not hard, nor computationally intensive since it can be performed via efficient hardware implementation of FFT.

From the sorted population of vectors we choose pairs in a top-to-bottom fashion. These pairs become parents

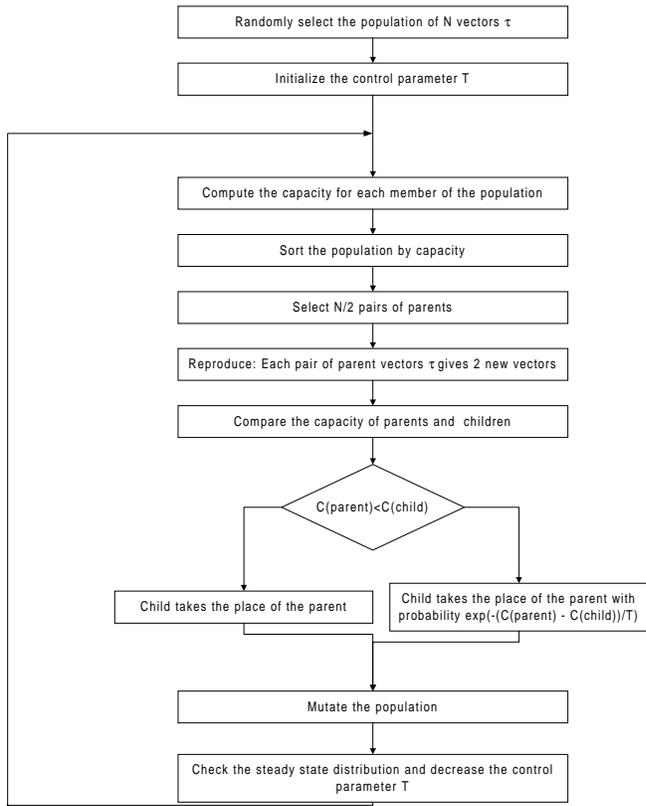


Figure 4: Flowchart of the hybrid filter design algorithm.

to new members of the population. Two new vectors are created from each set of parents by exchange of random number of bits whose position is selected at random as well. These new vectors are called children. The Boltzmann trial is held between the parents and children in following sense:

1. If the capacity of parent vector is smaller than the capacity of at least one of the children vectors, the better performing child vector takes the place of the parent.
2. If the capacity of parent vector is greater than the capacity of both child vectors, child vectors take the place of the parent with probability:

$$p = e^{-\frac{C(\tau_c) - C(\tau_p)}{T}} \quad (8)$$

We then proceed to mutate the population. This is performed by randomly selecting the pre-determined ratio of the population, and altering the random bits in the vector representation. Finally as the last step in the algorithm, we reduce the control parameter  $T$  in a logarithmic fashion.

### 3.2. Convergence Issues

From the description of the previous subsection it is obvious that the algorithm has strong similarities with the Simulated Annealing since it uses the probabilistic transition rule governed by the exponential of the difference between performances of the two candidates. The deviation from standard Simulated Annealing is twofold: there is  $N/2$  transitions happening in every iteration (generation) and neighborhood generation is dependent on the multiple members of the population.

For the proof of convergence we use idea of [10] to concatenate all vectors  $\tau$  in one large super-vector  $\xi$ . We define the performance criteria of this supervector as the sum of performances of individual vectors  $\tau$  it contains. The algorithm then becomes the Simulated Annealing algorithm of [11] over the extended space of supervectors  $\xi$  and asymptotic convergence of supervectors readily follows. This, in turn, implies convergence of the population to the singular distribution at the maximum capacity point.

## 4. EXPERIMENTS

In order to demonstrate the effectiveness of the proposed algorithm we have implemented the algorithm for the CSA6 9kft - 26ga copper wire test line. The transmitted signal power was -40dBm which is standard for the commercial DSL services. Additive noise was modeled as ETSIB. The algorithm was initialized with initial population of 5000 vectors, and run for 100 iterations (generations). The mutation factor was constant and equal to 20% of the population vectors and 20% of the bits in the mutated vectors. The mating was performed by exchanging 40 % of randomly selected bits from both parents and each parent was paired in the Boltzmann tournament with only the child closer in the Euclidean distance to reduce the computational load.

The results of the experiment are shown in the Figure 4. The proposed algorithm clearly outperformed the MMSE as early as generation 7 and converged at generation 40 to a value close to the maximum available capacity.

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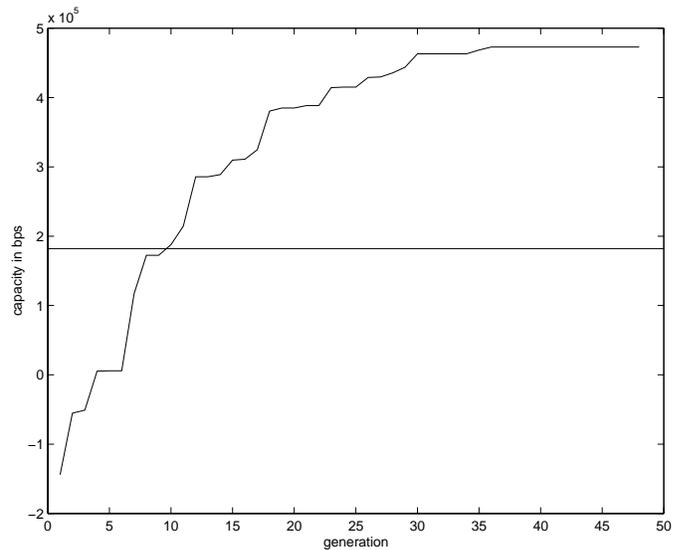


Figure 5: Comparison of the capacities achieved by MMSE equalizer design algorithm and the Simulated Annealing - Genetic Algorithm hybrid. Straight line represents the MMSE capacity which is constant with respect to iteration (generation) number. Increasing line represents the hybrid algorithm capacity which surpasses the MMSE capacity as early as 7<sup>th</sup> iteration, and rises to achieve near optimum capacity around generation 40.

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