IRREGULAR SAMPLING WITH UNKNOWN LOCATIONS

Pina Marziliano[1] *and Martin Vetterli*[1,2]

[1] LCAV, Ecole Polytechnique Fédérale de Lausanne, Switzerland
[2] EECS Dept., University of California at Berkeley, USA
email: [marziliano,vetterli]@de.epfl.ch

ABSTRACT

This paper is concerned with finding the locations of an irregularly sampled finite discrete-time band-limited signal. First a geometrical approach is described and is transformed into an optimization problem. Due to the structure of the problem, multiple solutions exist and are shifts of each other. Three methods of solution are suggested: an exhaustive method which finds the exact set of locations; random search method and cyclic coordinate method, both descent methods, which find approximate or exact solutions. The cyclic coordinate method is less likely to fall in a local minimum and proves to be more satisfactory than the random search method in the presence of jitter. A practical example, where a signal is sampled several times with a regular spacing, is also described.

1. INTRODUCTION

Irregular sampling appears in many practical applications. For instance, when sending data through a channel, if the channel is noisy , only a non-uniform subset of data is received. Under certain conditions, irregular sampling methods may recover the missing data.

Irregular sampling for band-limited (BL) signals have been extensively studied,(e.g.[3], [5]). The methods are iterative and consist in reconstructing a BL signal given the band-limit, the subsample values and the irregular set of locations. In some situations, for instance when there is jitter in the data, the locations of the irregularly sampled signal are unknown and the methods proposed by [3] may not be applied until these locations are found.

In this paper, we consider irregularly sampled discrete-time band-limited finite signals $(\ell_2(0, N-1)_{BL})$ with *unknown locations*. First we explore the geometry of the problem and derive the solution using a subspace approach, which is translated algebraically in order to solve the problem numerically. We show the existence of a solution in the discrete-time case. Methods for finding the unknown locations are proposed : exhaustive search method, random search method, cyclic coordinate method. The performance of the three algorithms are compared. Then, two practical examples are considered:

(a) when there is a small jitter in the sampling locations and

(b) when there are a few fixed shifted grids.

2. DEFINING THE PROBLEM

The discrete-time irregular sampling with unknown locations problem is as follows: Let $\mathbf{x} = [x_0, \ldots, x_{N-1}]^T$ be a discrete-time signal of length N. Then:

Definition 2.1 $\mathbf{x}$ *is* $L - BL$ *if there are* L *non-zero Fourier frequencies in the spectrum:*

$$\mathbf{X} = \mathbf{F}\mathbf{x} = [X_0, X_1, \ldots, X_{L-1}, 0, \ldots, 0]^T \qquad (1)$$

where $\mathbf{F}$ *is the* $N \times N$ *DFT matrix, defined by* $\{F_{nm} = W^{(n-1)(m-1)}\}_{n,m=1,\ldots,N}$ *and* $W = e^{-i2\pi/N}$.

Suppose K samples of $\mathbf{x}$ are observed, with

$$2 \le L \le K < N$$

$$\mathbf{x_n} = [x_{n_1}, x_{n_2}, \ldots, x_{n_K}]^T. \qquad (2)$$

The problem is to determine the locations:

$$\mathbf{n} = [n_1, n_2, \ldots, n_K], 0 \le n_i < n_j \le N-1, i < j. \qquad (3)$$

Denote this as the (N, K, L) problem where N is the length of the discrete time signal, K is the number of unknown locations from which we have sample values and L is the band-limit.

3. GEOMETRY OF THE (N, K, L) PROBLEM

In this section, we prove the existence and multiplicity of the solutions to the (N, K, L) problem. Based on a subspace approach, we derive conditions that enable to verify if a set of locations is a solution to the (N, K, L) problem.

3.1. Subspace approach

Consider $\mathbf{x}$ an L-BL signal and $\mathbf{x_n}$ a subsample of $\mathbf{x}$. From Definition 2.1, the spectrum of $\mathbf{x}$ has L non-zero components. We deduce that $\mathbf{x}$ belongs to the subspace spanned by the

L first columns of the inverse DFT, $\mathbf{F}^{-1}$. Denote the latter subspace by

$$\mathcal{V}_L = span\{\mathbf{f}_l\}_{l=0}^{L-1} \tag{4}$$

where $\mathbf{f}_l$ is the l^{th} column of $\mathbf{F}^{-1}$. Define $\mathcal{S}_K(\mathbf{n})$ as the subspace spanned by the canonical base vectors corresponding to the locations $\mathbf{n} = [n_1, n_2, \ldots, n_K]$ of the subsample $\mathbf{x_n}$,

$$\mathcal{S}_K(\mathbf{n}) = span\{\mathbf{e}_{n_i}\}_{i=1}^{K} \tag{5}$$

where $\mathbf{e}_j$ is an $1 \times N$ vector, with value 1 in $j-th$ position and 0 elsewhere. Hence,

$$\mathbf{x_n} \in \mathcal{P}(\mathbf{n}) = Proj_{\mathcal{S}_K(\mathbf{n})}\mathcal{V}_L \tag{6}$$

and $\mathbf{n}$ is a solution to the (N, K, L) problem.
A small example will illustrate .

Example 3.1 *Consider the following* $(4, 3, 2)$ *problem. Suppose* $\mathbf{x} = [x_0, x_1, x_2, x_3]^T$ *is a 2-BL signal.*

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W^{-1} & W^{-2} & W^{-3} \\ 1 & W^{-2} & W^{-4} & W^{-6} \\ 1 & W^{-3} & W^{-6} & W^{-9} \end{bmatrix} \cdot \begin{bmatrix} X_0 \\ X_1 \\ 0 \\ 0 \end{bmatrix} \in \mathcal{V}_2.$$

Let $\mathbf{n} = [n_1, n_2, n_3]$ *be the unknown locations. Then*

$\mathcal{S}_3(\mathbf{n}) = span\{\mathbf{e}_{n_1}, \mathbf{e}_{n_2}, \mathbf{e}_{n_3}\}$ *and* $\mathbf{x_n} \in \mathcal{P}(\mathbf{n}) = Proj_{\mathcal{S}_3(\mathbf{n})}\mathcal{V}_2$

The possible locations are:

$$\mathbf{m}^1 = [0, 1, 2], \mathbf{m}^2 = [0, 1, 3], \mathbf{m}^3 = [0, 2, 3], \mathbf{m}^4 = [1, 2, 3].$$

If $\mathbf{n} = [0, 1, 2]$ *is a solution then* $\mathbf{x}_{\mathbf{m}^1} \in \mathcal{P}(\mathbf{n}), \mathbf{x}_{\mathbf{m}^2} \notin \mathcal{P}(\mathbf{n})$ *and* $\mathbf{x}_{\mathbf{m}^3} \notin \mathcal{P}(\mathbf{n})$.

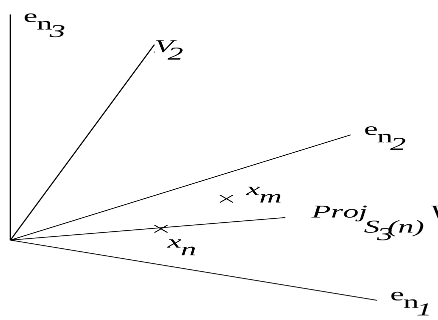

Figure 1: Geometrical interpretation of the (N,K,L) problem

Due to the shift property of the DFT, there may be multiple solutions to the (N, K, L) problem which are shifts of each other.

Theorem 3.1 *Let* $\mathbf{x_n}$ *be a subsample of an* $L - BL$ *signal* $\mathbf{x}$. *If* $\mathbf{n} = [n_1, n_2, \ldots, n_K]$ *is a solution to the* (N, K, L) *problem then all shifts* i *of* $\mathbf{n}$,

$$\mathbf{m} = \mathbf{n} + i = [n_1 + i, n_2 + i, \ldots, n_K + i]$$

are also solutions,with $1 \leq i < N - n_K$.

Proof: If $\mathbf{n} = [n_1, n_2, \ldots, n_K]$ is a solution of the (N, K, L) problem then $\mathbf{x_n} \in Proj_{\mathcal{S}_K(\mathbf{n})}\mathcal{V}_L$,where

$$\mathcal{V}_L = span\{\mathbf{f}_l = [1, W^{-l}, W^{-2l}, \ldots, W^{-(N-1)l}]\}_{l=0}^{L-1}.$$

The projection of $\mathcal{V}_L$ on $\mathcal{S}_K(\mathbf{n})$ is spanned by

$$\{(W^{-n_1 l}, W^{-n_2 l}, \ldots, W^{-n_K l})\}_{l=0}^{L-1}. \tag{7}$$

Similarly, the projection of $\mathcal{V}_L$ on the subspace spanned by the canonical base vectors corresponding to the shifted solution $\mathbf{m} = \mathbf{n} + i = (n_1 + i, n_2 + i, \ldots, n_K + i)$ is spanned by

$$\begin{aligned} & \{[W^{-(n_1+i)l}, W^{-(n_2+i)l}, \ldots, W^{-(n_K+i)l}]\}_{l=0}^{L-1} \\ = & \{W^{il}[W^{-n_1 l}, W^{-n_2 l}, \ldots, W^{-n_K l}]\}_{l=0}^{L-1}. \end{aligned} \tag{8}$$

Since equation (8) differs of equation (7) by a factor of W^{il}, this implies that $Proj_{\mathcal{S}_K(\mathbf{n})}\mathcal{V}_L = Proj_{\mathcal{S}_K(\mathbf{m})}\mathcal{V}_L$. Hence $\mathbf{m}$ is also a solution to the (N, K, L) problem. □

3.2. Algebraic form

In this section, we derive algebraic expressions which verify if a given set of subsample locations is a solution to the (N, K, L) problem.
Consider $\mathbf{n} = [n_1, n_2, \ldots, n_K]$ the set of locations associated to $\mathbf{x_n}$, a subsample of the $L - BL$ signal $\mathbf{x}$. From (1), $\mathbf{x}$ is fully recovered due to the non-singularity of $\mathbf{F}$,

$$\begin{aligned} \mathbf{x} &= \mathbf{F}^{-1}\mathbf{X} & (9) \\ &= \sum_{l=0}^{L-1} \mathbf{f}_l X_l & (10) \\ &= \mathbf{F}_L \mathbf{X}_L & (11) \end{aligned}$$

where $\mathbf{F}_L$ is the matrix formed by the columns $\{\mathbf{f}_l\}_{l=0}^{L-1}$ of $\mathbf{F}^{-1} = \frac{1}{N}\mathbf{F}^*$ and $\mathbf{X}_L = [X_0, X_1, \ldots, X_{L-1}]^T$ are the non-zero components of the spectrum $\mathbf{X}$. Hence $\mathbf{x}$ belongs to $\mathcal{V}_L$.
Furthermore, note that $\mathbf{x_n}$ is simply composed of the rows $\mathbf{n} = [n_1, n_2, \ldots, n_K]$ of $\mathbf{x}$. It is obtained by multiplying equation (11) on the left by a matrix $\mathbf{P_n} = [\mathbf{e}_{n_1}\mathbf{e}_{n_2}\ldots\mathbf{e}_{n_K}]^T$, where $\mathbf{P_n}$ is a projection matrix which projects $\mathbf{x}$ onto $\mathbf{x_n}$ and $\mathbf{e}_i$ is the $N \times 1$ canonical base vector. Hence,

$$\begin{aligned} \mathbf{x_n} &= \mathbf{P_n x} & (12) \\ &= \mathbf{P_n F}_L \mathbf{X}_L & (13) \end{aligned}$$

A priori, $\mathbf{X}_L$ is unknown and therefore an expression which gives $\mathbf{X}_L$ in terms of the observed values, $\mathbf{x_n}$, is desired. Define $\mathbf{M_n} = \mathbf{P_n F}_L$, the matrix composed of rows $\{n_i\}_{i=1}^K$ and first L columns of $\mathbf{F}^{-1}$. Then,

$$\begin{aligned} \mathbf{x_n} &= \mathbf{M_n X}_L & (14) \\ \mathbf{M_n^* x_n} &= \mathbf{M_n^* M_n X}_L & (15) \\ \Rightarrow \mathbf{X}_L &= (\mathbf{M_n^* M_n})^{-1}\mathbf{M_n^* x_n}. & (16) \end{aligned}$$

$(\mathbf{M}_\mathbf{n}^* \mathbf{M}_\mathbf{n})^{-1}$ is the generalized inverse and exists since the columns of $\mathbf{F}_L$ are linearly independent.
Hence to verify if a set of locations $\mathbf{m} = [m_1, m_2, \ldots, m_K]$ is a solution to the (N, K, L) problem we must verify that $\mathbf{x_n} \in Proj_{\mathcal{S}_K(\mathbf{m})}\mathcal{V}_L$. Algebraically, this is equivalent to verifying that

$$\mathbf{x_m} = \mathbf{M_m}(\mathbf{M_m^*}\mathbf{M_m})^{-1}\mathbf{M_m^*}\mathbf{x_n} \tag{17}$$

is equal to $\mathbf{x_n}$ or equivalently,

$$(\mathbf{M_m}(\mathbf{M_m^*}\mathbf{M_m})^{-1}\mathbf{M_m^*} - \mathbf{I})\mathbf{x_n} = 0. \tag{18}$$

The latter gives a sufficient condition for the existence of a solution. A closed form formula for the solution is obtained by solving the system of K nonlinear equations in (18) with respect to the K unknowns, $\mathbf{m} = [m_1, m_2, \ldots, m_K]$. Since the system is nonlinear, it may admit more than one solution.[1]

4. SOLVING METHODS

In this section, we present three methods to numerically solve the (N, K, L) problem. The optimality of a solution $\mathbf{m}$ is tested by verifying if the ℓ^2-norm of equation (18)

$$E(\mathbf{m}) = ||(\mathbf{M_m}(\mathbf{M_m^*}\mathbf{M_m})^{-1}\mathbf{M_m^*} - I)\mathbf{x_n}|| \tag{19}$$

equals zero for the Exhaustive method or $E(\mathbf{m})$ is minimized for the Random Search and Cyclic Coordinate methods.

4.1. Exhaustive Method

An elementary way to solve the (N, K, L) problem is using an exhaustive search approach. This method consists in verifying equation (19) for all $\binom{N}{K}$ sets of locations. From Theorem 3.1 since some sets of locations are just shifts of each other, we put these sets in one class and elect a representative for each class. The representative satisfying equation (19) with $E(\mathbf{m}) = 0$ is a solution of the (N, K, L) problem.

4.2. Random search (RS) and Cyclic coordinate (CC) method

The random search method is an iterative descent algorithm. A component of the location set is perturbed according to a probability distribution. If the cost function ($E(\mathbf{m})$) of the perturbed solution decreases, we keep perturbing the same component. Otherwise we perturb another component and continue this way until a global or local minimum is found.

Algorithm 4.1 Random search method
Initial Step:

Choose an initial set of K locations $\mathbf{m}^1$, and let $\mathbf{m} = \mathbf{m}^1$, $j = 1$. Go to general step.

General Step:

1. *Obtain $\mathbf{m}^2$ by perturbing component j of $\mathbf{m}$ by $\{+1, -1\}$ with probability 0.5. Go to step 2.*
2. *If $E(\mathbf{m}^2) = 0$ then $\mathbf{m} = \mathbf{m}^2$. Go to step 3.*
   If $|E(\mathbf{m}^2) - E(\mathbf{m})| = 0$, then go to step 3.
   If $E(\mathbf{m}^2) < E(\mathbf{m}^1)$ then $\mathbf{m} = \mathbf{m}^2, \mathbf{m}^1 = \mathbf{m}^2$ and repeat step 1. Otherwise, $j = j + 1, \mathbf{m} = \mathbf{m}^1$. If $j \leq K$ repeat step 1.
3. *If $E(\mathbf{m}) = 0$ then $\mathbf{m}$ is global minimum. Otherwise it is a local one. Stop.*

The cyclic coordinate method is similar to the gradient descent but does not require any derivative information. It uses the coordinate direction axis as the search directions. It differs from the Random search method in that the perturbation is not probabilistic but deterministic.

Algorithm 4.2 Cyclic coordinate method
Initial Step:

Let $\mathbf{d}_1, \ldots, \mathbf{d}_K$ be the coordinate directions, where $\mathbf{d}_i$ is $1 \times K$ vector with value 1 in position i, and 0 elsewhere. Choose an initial set of K locations $\mathbf{m}^1$, and let $\mathbf{m} = \mathbf{m}^1$, $j = 1$. Go to general step.

General Step:

1. *$\mathbf{m}$ is obtained by perturbing component j of $\mathbf{m}^1$ by*
   $$\lambda_j = arg \min_{\lambda \in \{-1,0,1\}} \{E(\mathbf{m}^1 + \lambda \mathbf{d}_j)\},$$
   $\mathbf{m} = \mathbf{m}^1 + \lambda_j \mathbf{d}_j$. Go to step 2.
2. *If $E(\mathbf{m}) = 0$ then go to step 3.*
   If $|E(\mathbf{m}^1) - E(\mathbf{m})| = 0$, then go to step 3.
   Otherwise $j = j + 1, \mathbf{m}^1 = \mathbf{m}$. If $j \leq K$ then repeat step 1. If $j > K$ then $j = 1$ and repeat step 1 .
3. *If $E(\mathbf{m}) = 0$ then $\mathbf{m}$ is a global minimum. Otherwise it is a local one. Stop.*

Both methods do not guarantee a global minimum. When stuck in a local minimum, a random set of locations is generated and the methods are repeated. The tested algorithm comes to a halt when a global minimum is obtained or an upper bound on the number of local minimum is exceeded.

5. EXPERIMENTS

In this section, we do some experiments on the algorithms proposed in Section 4. All methods need to calculate the value of $E(\mathbf{m})$ which involves matrix multiplication and matrix inverse operations on matrices of size $K \times L$, $K \times K$, respectively.
The exhaustive search method is certain to find a solution. It is most time consuming when the number of unknown locations K is close to $N/2$ which is due to its combinatorial nature.

[1] If $K = L$ then there is an infinite number of solutions. This is due to the fact that the equivalent condition of existence is the identity, $\mathbf{M_m}\mathbf{M_m^{-1}} = I$, where $\mathbf{M_m}$ is a square matrix.

5.1. Subsample locations with jitter

As mentioned in the introduction, the importance of finding the unknown locations is due to the presence of jitter in the data. Jitter occurs when the sample values location is mistaken for another location: $x_{n_i} = x_{n_i+j}$, where j is the jitter and the n_i are uniformly spaced. Different types of jitter are described in [2]. We applied the RS and CC methods on data with jitter around a correct value following a symmetric probability distribution,

$$P(j = \pm 1) = p, P(j = 0) = 1 - 2p.$$

This corresponds to taking samples around multiples of a sampling interval , but with a certain time location uncertainty. Figure 2 shows that the percentage of finding the correct locations decreases with the probability of the jitter. Also, the CC and RS methods find solutions to $(N, K, L) = (16, 4, 2)$ problem on average 82%, 52% of the time, respectively. Therefore the Cyclic Coordinate method is less likely to get stuck in a local minimum.

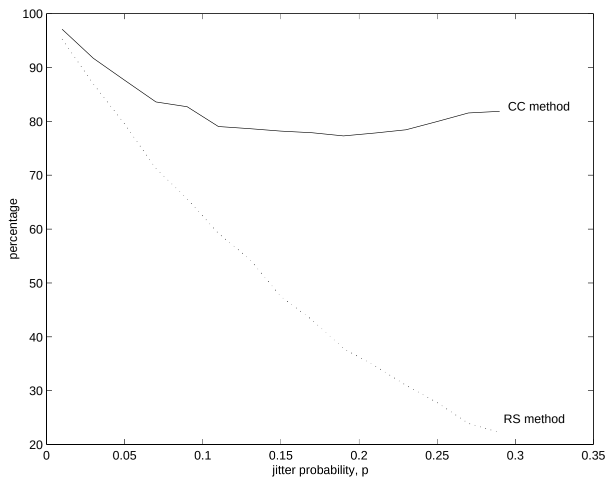

Figure 2: Percentage of finding a global minimum for CC and RS methods with symmetric probability distribution for jitter, $N = 16, K = 4, L = 2$, 100 simulations on 50 different signals.

5.2. Subsample locations with unknown shift

As a practical example, consider the following setup. Suppose two photographs of a scene are taken, where one is a shifted version of the other, but with unknown shift. Combining the two sets of data and verifying equation (19) for all possible shifts, the correct shift is determined and the image may be reconstructed using irregular sampling. For computational reasons, we ran a one-dimensional example to see how well the shift can be recovered. Suppose the signal is of length $N = 256$ and band-limited to $L < 64$. Take two sets of samples at locations $8n$ and $8n + k, 0 \leq n \leq 31, 1 \leq k \leq 7$, with shift k unknown. This gives an irregular set of $K = 64$ locations. Figure 3 shows $E(\mathbf{m})$ for k varying 1 to 7 (where the optimal shift is $k^* = 4$) and various $L, (33 \leq L \leq 63)$. Clearly, $k^* = 4$ can be recovered and $E(\mathbf{m})$ is least for L closer to K.

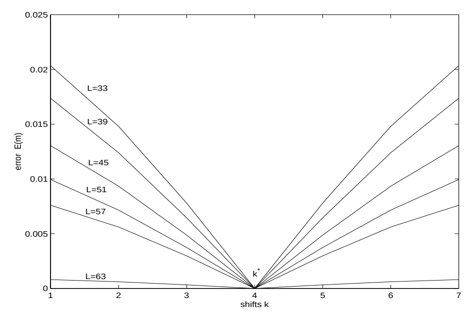

Figure 3: Finding the unknown shift, k, for signal length N=256, with varying band-limit $L, 33 \leq L \leq 63$.

6. CONCLUSION

We have treated an essential element of the irregular sampling problem, which is that of not knowing the locations of the subsamples. Most of the fast methods already developed assume the knowledge of these locations but in certain applications where there is jitter in the data these locations are unknown. The solving methods we described find these unknown locations.

7. REFERENCES

[1] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty. *Nonlinear Programming Theory and Algorithms*, Wiley, New York, 2nd Edition, 1993.

[2] W. M. Brown. Sampling with random jitter, *J. Siam*, **Volume**, 11, pp.460-473, 1963.

[3] H. G. Feichtinger and K. Grochenig. Theory and practice of irregular sampling,*Wavelets Mathematics and Applications*, pp.305-363, 1994.

[4] K. Grochenig. A discrete Theory of Irregular Sampling, *Linear Algebra and its applications*, **Volume**, 193, pp.129-150, 1993.

[5] T. Strohmer. Efficient Methods for Digital Signal and Image Reconstruction from Nonuniform Samples, Ph.D Thesis, University of Vienna, 1993.