REPRODUCING KERNEL STRUCTURE AND SAMPLING ON TIME-WARPED KRAMER SPACES

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ABSTRACT

Given a signal space of functions on the real line, a time-warped signal space consists of all signals that can be formed by composition of signals in the original space with an invertible real-valued function. Clark's theorem shows that signals formed by warping bandlimited signals admit formulae for reconstruction from samples. This paper considers time warping of more general signal spaces in which Kramer's generalized sampling theorem applies and observes that such spaces admit sampling and reconstruction formulae. This observation motivates the question of whether Kramer's theorem applies directly to the warped space, which is answered affirmatively by introduction of a suitable reproducing kernel Hilbert space structure. This result generalizes one of Zeevi, who pointed out that Clark's theorem is a consequence of Kramer's.

1. INTRODUCTION

Given a space S of signals $f : \mathbb{R} \to \mathbb{C}$ and an invertible function $\gamma : \mathbb{R} \to \mathbb{R}$, the *time-warped* signal space S_γ consists of all functions of the form $h = f \circ \gamma$. In the case that S is the space of B of Ω-bandlimited signals (i.e., functions of the form

$$f(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{f}(\omega) e^{i\omega t} \, d\omega \tag{1}$$

with $0 < \Omega < \infty$ and $f \in L^2(\mathbb{R})$), a result of J.J. Clark *et al.* [3] shows that the space B_γ of time-warped bandlimited signals admits the formula

$$h(t) = \sum_n h(\tau_n) \operatorname{sinc}\left[\frac{\Omega(\gamma(t) - nT)}{\pi}\right]$$

for reconstruction of h from samples $h(\tau_n) = h(\gamma^{-1}(nT))$.

In this expression, $T = \pi/\Omega$ is the so-called Nyquist interval for B and $\operatorname{sinc}(t) = \sin(\pi t)/\pi t$. The sampling times $\{\tau_n\}$ are generally nonuniformly spaced and B_γ contains signals that are not bandlimited [5], so Clark's result provides a means for reconstructing certain spaces of non-bandlimited signals from nonuniformly spaced samples.

In [11], Y.Y. Zeevi and E. Shlomot noted that Clark's theorem for time-warped bandlimited functions may be seen as a special case of Kramer's well known generalized sampling theorem [8]. On the other hand, it has been observed [4] that for any signal space S that admits a reconstruction formula of the form

$$f(t) = \sum_n f(t_n) \phi_n(t) \tag{2}$$

there is a sampling theorem with reconstruction formula

$$h(t) = \sum_n h(\gamma^{-1}(t_n)) \phi_n(\gamma(t)) \tag{3}$$

for time-warped signals in S_γ. Thus Clark's basic idea applies to time-warped signal spaces in addition to B_γ.

This paper considers time-warped signal spaces of the form K_γ where K is a "Kramer" space of signals for which Kramer's theorem yields a reconstruction formula of the form (2). When endowed with the appropriate inner product, K_γ is shown to admit a reproducing kernel and thus become a reproducing kernel Hilbert space (RKHS). The corresponding sampling theorem is, however, identical to the one obtained by applying Clark's method. Moreover, time-warped Kramer spaces with the RKHS inner product are seen to themselves be Kramer spaces and thus Clark's sampling theorem in these spaces is subsumed by Kramer's theorem, as in the case of time-warped bandlimited signals.

Unitary time warping of finite-energy (L^2) signals has received recent attention in connection with several signal processing applications [1, 2, 6]. This paper closes with extensions of the results developed to unitarily time-warped signal spaces.

2. CLASSICAL SAMPLING THEOREMS

The well known sampling theorem of Whittaker, Kotel'nikov, and Shannon (WKS) establishes that a bandlimited signal f of the form (1) can be reconstructed from uniformly spaced samples by the WKS formula

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)\,\mathrm{sinc}\left[\frac{\Omega(t-nT)}{\pi}\right] \qquad (4)$$

where $T = \pi/\Omega$ (as above) and the convergence is absolute [7]. To set the stage for the results in the following sections of this paper, this section summarizes Clark's and Kramer's extensions of this theorem. Since no generality is sacrificed, the remainder of the paper will assume $\Omega = \pi$ to make $T = 1$ and simplify the formulae presented.

2.1. Clark's theorem

If $h = f\circ\gamma$ (i.e., $f(t) = f(\gamma(t))$ for all $t \in \mathbb{R}$) with $f \in B$ and γ a warping function as described above, then defining $\tau_n = \gamma^{-1}(n)$ yields $h(\tau_n) = f(\gamma(\gamma^{-1}(n))) = f(n)$ so that the WKS formula (4) gives

$$f(t) = \sum_n h(\tau_n)\,\mathrm{sinc}[t-n]$$

and hence

$$h(t) = f(\gamma(t)) = \sum_n h(\tau_n)\,\mathrm{sinc}[\gamma(t)-n] \qquad (5)$$

With $\gamma(t) = t$, Clark's formula (5) reduces to (4) with $T = 1$. Moreover, for γ a non-affine function (i.e., $\gamma(t)$ is not of the form $at + b$ with a and b real numbers), the sampling times $\{\tau_n\}$ will generally be non-uniformly spaced and the space B_γ will contain non-bandlimited signals. Thus Clark's theorem generalizes the WKS theorem. Further analysis of the space B_γ is undertaken in [4] and [5].

2.2. Kramer's theorem

Kramer's generalized sampling theorem [8] considers signals supported in a bounded interval $\mathbf{I}$ and supposes the existence of a transform kernel $\psi : \mathbb{R}\times\mathbf{I} \to \mathbb{C}$ such that $\psi(t,\cdot) \in L^2(\mathbf{I})$ for each real t. The *Kramer space* K associated with $\mathbf{I}$ and ψ consists of all signals of the form

$$f(t) = \int_{\mathbf{I}} \psi(t,\omega)\tilde{f}(\omega)\,d\omega, \qquad (6)$$

with $\tilde{f} \in L^2(\mathbf{I})$. If there exists a countable set $\{t_n\} \subset \mathbb{R}$ such that $\{\psi(t_n,\cdot)\}$ is a complete orthogonal set on $L^2(\mathbf{I})$, then K admits the reconstruction formula

$$f(t) = \lim_{N\to\infty} \sum_{|n|\le N} f(t_n)s_n(t)$$

where

$$s_n(t) = \frac{\int_{\mathbf{I}} \psi(t,\omega)\overline{\psi(t_n,\omega)}\,d\omega}{\int_{\mathbf{I}} |\psi(t_n,\omega)|^2\,d\omega} \qquad (7)$$

With $\mathbf{I} = [-\Omega,\Omega]$, $\psi(t,\omega) = e^{i\omega t}/2\pi$, and $t_n = n\pi/\Omega$, Kramer's theorem reduces to the WKS theorem. Hence this result, like Clark's, generalizes the WKS result.

3. RKHS STRUCTURE ON WARPED KRAMER SPACES

Recall that a *reproducing kernel* (RK) on a Hilbert space $\mathcal{H}$ of complex-valued functions on $\mathbb{R}$ is a function $k : \mathbb{R}^2 \to \mathbb{C}$ such that $k(\cdot,x) \in \mathcal{H}$ for each real x and $f(x) = \langle f, k(\cdot,x)\rangle$ for every $x \in \mathbb{R}$ and $f \in \mathcal{H}$.

Let K be a Kramer space and let γ be a warping function. To show that K_γ admits a RKHS structure, note that (6) implies each $f_\gamma = f\circ\gamma \in K_\gamma$ has a representation

$$f_\gamma(t) = \int_{\mathbf{I}} \psi(\gamma(t),\omega)\tilde{f}(\omega)d\omega \qquad (8)$$

Define an inner product $\langle\cdot,\cdot\rangle$ in K_γ by

$$\langle f_\gamma, g_\gamma\rangle = \int_{\mathbf{I}} \tilde{f}(\omega)\overline{\tilde{g}(\omega)}\,d\omega \qquad (9)$$

and $k_\gamma(t,x)$ by

$$k_\gamma(t,x) = \int_{\mathbf{I}} \psi(\gamma(t),\omega)\overline{\psi(\gamma(x),\omega)}d\omega \qquad (10)$$

Comparing (10) with (8) shows that $k_\gamma(\cdot,x)$ is the integral transform of $\overline{\psi(\gamma(x),\cdot)}$ and hence

$$\langle f_\gamma, k_\gamma(\cdot,x)\rangle = \int_{\mathbf{I}} \tilde{f}(\omega)\psi(\gamma(x),\omega)d\omega = f_\gamma(x) \qquad (11)$$

Thus k_γ is a RK for K_γ.

3.1. RKHS structure and sampling

Clark's observation shows that the warped Kramer space K_γ admits a reconstruction formula with sampling times $\tau_n = \gamma^{-1}(t_n)$ and interpolation functions $s_n(\gamma(t))$ obtained from the s_n defined in (7). Using the RKHS structure on K_γ defined above allows this to be deduced both as a direct consequence of Kramer's theorem (i.e., without reference to Clark's approach) and as a corollary to a standard result about sampling formulae in RKHS. With $K = B$, the first of these results reduces to confirm Zeevi and Shlomot's remark about Clark's theorem following from Kramer's.

Let $\{t_n\}$ be a sampling set for K and define $\phi(t,\omega) = \psi(\gamma(t),\omega)$. With $\{\tau_n\}$ as defined above, the facts that $\{\psi(t_n,\cdot)\}$ is a complete orthogonal set in $L^2(\mathbf{I})$ and

$$\{\phi(\tau_n,\omega)\} = \{\psi(t_n,\omega)\}$$

imply that $\{\phi(\tau_n,\cdot)\}$ is a complete orthogonal set in $L^2(\mathbf{I})$. Moreover, with this notation equation (8) shows that

$$f_\gamma(t) = \int_{\mathbf{I}} \phi(t,\omega)\tilde{f}_\gamma(\omega)\,d\omega$$

for each $f_\gamma \in K_\gamma$. Hence Kramer's theorem allows reconstruction of f_γ from samples at $\{f_\gamma(\tau_n)\}$ by

$$f_\gamma(t) = \sum_n f_\gamma(\tau_n)s_n^\gamma(t)$$

with

$$\begin{aligned} s_n^\gamma(t) &= \frac{\int_I \psi(\gamma(t),\omega)\overline{\psi(\gamma(\tau_n),\omega)}d\omega}{\int_I |\psi(\gamma(\tau_n),\omega)|^2 d\omega} \qquad (12) \\ &= \frac{\int_I \psi(\gamma(t),\omega)\overline{\psi(t_n,\omega)}d\omega}{\int_I |\psi(t_n,\omega)|^2 d\omega} \\ &= s_n(\gamma(t)) \end{aligned}$$

exactly as defined by Clark's observation.

The relationship between sampling and reproducing kernels is well established [9, 10]. In particular, a sampling basis $\{v_n\}$ of a RKHS yields a reconstruction formula

$$f(t) = \sum_n f(t_n)v_n(t)$$

for a sampling set $\{t_n\}$ if and only if its biorthogonal basis $\{V_n\}$ is given by

$$V_n(x) = \langle V_n, v_n\rangle\, k(t_n,x) \qquad (13)$$

Recall that two sets $\{v_n\}$ and $\{V_n\}$ are biorthogonal if $\langle v_n, V_m\rangle = \delta_{nm}\langle v_n, V_n\rangle$. Comparing (12) with (8) shows that $s_n^\gamma(t)$ is the integral transform of

$$\frac{\overline{\psi(\gamma(\tau_n),\omega)}}{\int_I |\psi(\gamma(\tau_n),\omega)|^2 d\omega}$$

and hence

$$\langle s_n^\gamma, k_\gamma(t_m,\cdot)\rangle = \delta_{nm}$$

when the inner product is as defined in (9). Therefore the biorthogonal basis of the sampling basis $\{s_n^\gamma\}$ arises from the RK, (13) is satisfied, and the sampling basis $\{v_n\}$ is identical to $\{s_n^\gamma\}$.

4. SAMPLING IN UNITARILY WARPED SPACES

As mentioned earlier, the role of unitary operators in signal processing has received considerable attention in recent years in connection with several applications. If γ is a differentiable warping function, the mapping taking $f \in L^2(\mathbb{R})$ to the signal h with values $h(t) = \sqrt{|\gamma'(t)|}f(\gamma(t))$ is easily verified to be a unitary operator on $L^2(\mathbb{R})$.

The machinery and results developed in the previous section of this paper extend readily to unitarily warped Kramer spaces. In particular, the reproducing kernel becomes

$$k_\gamma(t,x) = \sqrt{|\gamma'(x)\gamma'(t)|}\int_{\mathbf{I}} \psi(\gamma(t),\omega)\overline{\psi(\gamma(x),\omega)}d\omega$$

and, assuming no $\gamma'(t_n) = 0$, the sampling basis

$$v_n(t) = \sqrt{\left|\frac{\gamma'(t)}{\gamma'(t_n)}\right|}s_n(\gamma(t))$$

and Parseval-like relationship

$$\sum_n \frac{|h(t_n)|^2}{\sqrt{|\gamma'(t_n)|}} = \int |h(t)|^2 dt$$

are obtained.

5. CONCLUSION

This paper has shown that warped and unitarily warped Kramer spaces admit RKHS structures, in view of which three perspectives apply to yield sampling theorems for such spaces. Clark's perspective was seen to be subsumed by Kramer's theorem in this setting, as has been pointed out by other authors in the special case of time-warped bandlimited signals. Furthermore, the introduction of a RKHS structure allows the use of standard results on sampling in RKHS to obtain the sampling theorems generated by Clark's and Kramer's machinery in these time-warped spaces.

6. REFERENCES

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