A NEW CLASS OF AFFINE HIGHER ORDER TIME-FREQUENCY REPRESENTATIONS*

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ABSTRACT

We propose a *new* class of affine higher order time-frequency representations (HO-TFRs) unifying HO-TFRs which satisfy the desirable properties of scale covariance and time-shift covariance. This new class extends to higher order ($N > 2$) the affine class of quadratic ($N = 2$) time-frequency representations. In this paper, we provide five alternative formulations of the class in terms of multi-dimensional smoothing kernels. We discuss important class members, including the *new* higher order scalogram that is related to the wavelet transform. We also list additional desirable properties and derive the associated kernel constraints. Finally, we consider a subclass of affine HO-TFRs that intersects with a Cohen's class of time and frequency shift covariant HO-TFRs. A formulation for HO-TFRs satisfying three covariances in this higher order affine-Cohen intersection is derived.

1. INTRODUCTION

With the advent of the well-known wavelet transform [12] much attention has been given to signal representations that preserve scale changes (i.e. compressions and dilations) on the analysis signal. Such representations are referred to as scale covariant [6]. The scale covariance property has been exploited in fractal signal analysis (e.g. the estimation of local scaling exponents [3]), and in wideband Doppler signals (e. g. the detection and estimation of wideband signals [3]). Scale covariance is one of the fundamental properties of the affine class of quadratic time-frequency representations (QTFRs) [2, 3, 6, 13]. Specifically, all affine class QTFRs preserve scale changes as well as time shifts on the analysis signal.

Any affine QTFR can be expressed as an affine smoothing of the quadratic Wigner distribution (WD) with a two-dimensional kernel that uniquely characterizes the QTFR [13]. For example, the scalogram (squared magnitude of the wavelet transform) is an important affine QTFR whose kernel is the WD of the wavelet function. The scalogram is useful in applications requiring "constant-Q" analysis (frequency dependent resolution in the time-frequency plane), such as the analysis of short duration transient signals. Other desirable properties of affine QTFRs can be given in terms of constraints on the two-dimensional kernel [2, 3, 6].

Another important QTFR class defined in terms of smoothing kernels is Cohen's class of constant time and frequency shift covariant QTFRs [1, 2, 6]. Those affine class QTFRs which also preserve frequency shifts, such as the WD, are also members of Cohen's class. They form the affine-Cohen intersection subclass [7] satisfying (i) time-shift covariance, (ii) frequency-shift covariance and (iii) scale covariance.

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In [4, 5, 16], the quadratic WD and Cohen's class were extended to higher order. These higher order Cohen's classes were obtained by smoothing a higher order WD (HO-WD) with a multi-dimensional kernel. Some higher order Cohen's classes are covariant only to time shifts [4, 5], others to both time and frequency shifts [9]. Members include the HO-WD and higher order versions of the spectrogram and the Choi-Williams distribution [4, 5]. A version of the quadratic hyperbolic class of scale covariant and hyperbolic time-shift covariant QTFRs [11] was also extended to higher order [9, 14]. Thus, there currently exist higher order versions of Cohen's class and of the hyperbolic class, but to our knowledge, there is no higher order extension of the affine class.

In this paper, we propose a *new* higher order extension of the quadratic affine class which provides a unifying framework for HO-TFRs that satisfy the important properties of scale covariance and time-shift covariance. We provide five *new* alternative "normal form" [2, 6] expressions for affine HO-TFRs in terms of multi-dimensional kernel functions. We list important members of the higher order affine class, which include the *new* higher order scalogram. In addition to scale covariance and time-shift covariance, we present additional desirable properties and derive their corresponding kernel constraints. Finally, we provide a simplified formulation in terms of a one-dimensional function for HO-TFRs in the intersection of the higher order affine class and higher order Cohen's class. This extends to higher order the intersection of the quadratic affine class and Cohen's class [7]. Members of this higher order intersection preserve time shifts, frequency shifts and scale changes on the analysis signal (see Figure 1).

2. NEW HIGHER ORDER AFFINE CLASS

The second order affine class [2, 3, 6, 13] consists of all QTFRs, $A_X(t, f)$, of a signal $x(t)$ with Fourier transform (FT) $X(f)$, that satisfy the scale covariance property and the time-shift covariance property defined, respectively, as:

$$Y(f) = |a|^{\frac{1}{2}} X(af) \Longrightarrow A_Y(t, f) = A_X(\frac{t}{a}, af), \forall a \in \mathbb{R} \qquad (1)$$

$$Y(f) = e^{-j2\pi\tau f} X(f) \Longrightarrow A_Y(t, f) = A_X(t-\tau, f), \forall \tau \in \mathbb{R}. \qquad (2)$$

Any affine QTFR can be expressed as an affine convolution [13] of the WD of the signal, $\mathrm{WD}_X(t, f)$, with a two-dimensional kernel, $\psi_A(c, b)$, that uniquely characterizes the affine QTFR[1]:

$$A_X(t, f) = \int_{t'}\int_{f'} \psi_A(f(t-t'), \frac{-f'}{f}) \mathrm{WD}_X(t', f') dt' df'. \qquad (3)$$

[1]Unless otherwise specified, integration limits are -∞ to ∞.

We seek a higher order affine class of HO-TFRs that satisfy higher order extensions of (1) and (2). In particular, we want to derive the class of multi-time/uni-frequency Nth order HO-TFRs, $A_X^N(t_1,\cdots,t_{N-1},f)$, that satisfy:

$$Y(f)=|a|^{\frac{N-1}{N}}X(af) \Longrightarrow A_Y^N(t_1,\cdots,t_{N-1},f)= A_X^N(\frac{t_1}{a},\cdots,\frac{t_{N-1}}{a},af), \forall a\in\mathbb{R}, N\in\mathbb{N} \quad (4)$$

$$Y(f)=e^{-j2\pi\tau f}X(f) \Longrightarrow A_Y^N(t_1,t_2,\cdots,t_{N-1},f)= A_X^N(t_1-\tau,t_2+\tau,\cdots,t_{N-1}-\tau,f), \forall\tau\in\mathbb{R}, N\text{even}. \quad (5)$$

For $N=2$, (4) and (5) simplify to (1) and (2), respectively.

Our new Nth order extension of the quadratic affine class is based on the multi-time/uni-frequency[2] HO-WD [15]:

$$\text{WD}_X^N(t_1,\cdots,t_{N-1},f)= \int_{\nu_1}\!\!\cdots\!\int_{\nu_{N-1}} V_X^N(f,\nu_1,\cdots,\nu_{N-1})\prod_{n=1}^{N-1}e^{j2\pi t_n\nu_n}d\nu_n \quad (6)$$

where the Nth order spectral product is $V_X^N(f,\nu_1,\cdots,\nu_{N-1})= X^*(f-\frac{1}{N}\sum_{i=1}^{N-1}\nu_i)\prod_{n=1}^{N-1}(\mathcal{L}_nX)(f+\nu_n-\frac{1}{N}\sum_{i=1}^{N-1}\nu_i)$, and the conjugation operator $\mathcal{L}_n$ conjugates the spectrum if the index n is even, e.g. $(\mathcal{L}_4X)(f)=X^*(f)$. The uni-time/multi-frequency HO-WD always satisfies the scale covariance in (4); it also satisfies the alternating sign time-shift covariance in (5) for N even.

We propose a multi-time/uni-frequency higher order affine class formulation that is a multi-dimensional affine convolution of the HO-WD in (6) with an N-dimensional kernel, $\psi_A^N(c_1,\cdots,c_{N-1},b)$:

$$A_X^N(t_1,\cdots,t_{N-1},f)=|f|^{N-2}\int_{f'}\int_{t_1'}\cdots\int_{t_{N-1}'}\text{WD}_X^N(t_1',\cdots,t_{N-1}',f') \cdot\psi_A^N\left(f(t_1-t_1'),\cdots,f(t_{N-1}-t_{N-1}'),\frac{-f'}{f}\right)df'\prod_{n=1}^{N-1}dt_n'. \quad (7)$$

As desired, (7) simplifies to the quadratic affine class in (3) when $N=2$. The resulting affine HO-TFRs in (7) always satisfy the desirable properties in (4) and (5). The HO-WD is a member of the new higher order affine class with kernel $\psi_{\text{WD}}^N(c_1,\cdots,c_{N-1},b)$ $=\delta(b+1)\prod_{i=1}^{N-1}\delta(c_i)$. We obtain *new* HO-TFRs such as the higher order scalogram by appropriately choosing the kernel function ψ_A^N in (7) (see section 2.1).

In Table 1, we propose four additional normal forms [2, 6] equivalent to any affine HO-TFR in (7). These alternative forms offer additional insight and often computational advantage. Normal forms A-I and A-II are given in terms of the signal, $x(t)$, and its spectrum, $X(f)$, respectively. Normal form A-III repeats (7). Normal form A-IV is in terms of a higher order ambiguity function. Normal form A-V is a version of A-II with simplified spectral arguments and is the Nth order version of the second order "bi-frequency" form [2]. The kernels ϕ_A^N, Φ_A^N, ψ_A^N and Ψ_A^N in normal forms A-I – A-IV, respectively, are interrelated by multi-dimensional FTs:

$$\phi_A^N(c_1,\cdots,c_{N-1},\zeta)=\int_b\psi_A^N(c_1,\cdots,c_{N-1},b)e^{j2\pi b\zeta}db$$
$$=\int_b\int_{\beta_1}\!\cdots\!\int_{\beta_{N-1}}\Phi_A^N(b,\beta_1,\cdots,\beta_{N-1})e^{j2\pi b\zeta}db\prod_{n=1}^{N-1}e^{j2\pi c_n\beta_n}d\beta_n$$
$$=\int_{\beta_1}\!\cdots\!\int_{\beta_{N-1}}\Psi_A^N(\zeta,\beta_1,\cdots,\beta_{N-1})\prod_{n=1}^{N-1}e^{j2\pi c_n\beta_n}d\beta_n. \quad (8)$$

The kernels in normal forms A-V and A-II are related as: $\Gamma_A^N(b_0,b_1,\cdots,b_{N-1})=\Phi_A^N(\frac{-1}{N}\sum_{k=0}^{N-1}b_k,b_1-b_0,\cdots,b_{N-1}-b_0)$.

2.1. Higher order affine class members

The quadratic scalogram (SCAL) is defined as the squared magnitude of the well-known wavelet transform (WT) [12]:

$$\text{SCAL}_X(t,f)=|\text{WT}_X(t,f)|^2=WT_X^*(t,f)WT_X(t,f) \quad (9)$$

where the WT is defined as:

$$\text{WT}_X(t,f)=\sqrt{\left|\frac{f}{f_r}\right|}\int_{t'}x(t')h^*\left(\frac{f}{f_r}(t'-t)\right)dt'. \quad (10)$$

Here, $x(t)$ is the input signal, $h(t)$ is the wavelet function and $f_r>0$ is a reference frequency. The scalogram kernel in (3) is $\psi_{\text{SCAL}}(c,b)=\text{WD}_H(-c/f_r,-f_rb)$, where $H(f)$ is the FT of $h(t)$. Since the scalogram is a correlation between the signal and the dilated/compressed wavelet, it implements a multi-resolution analysis.

We propose the higher order scalogram (HO-SCAL) as a member of the higher order affine class in (7) whose kernel is given by $\psi_{\text{SCAL}}^N(c_1,\cdots,c_{N-1},b)=\text{HO-WD}_H^{N*}(\frac{-c_1}{f_r},\cdots,\frac{-c_{N-1}}{f_r},-f_rb)$. Thus, the HO-SCAL is a multi-dimensional affine convolution of the HO-WD of the input signal with the HO-WD of the wavelet. Substituting ψ_{SCAL}^N into (7), we obtain the higher order extension of the scalogram:

$$\text{SCAL}_X^N(t_1,\cdots,t_{N-1},f)= |f_rf|^{\frac{N}{2}-1}WT_X^*\left(\sum_{i=1}^{N-1}t_i,f\right)\prod_{n=1}^{N-1}(\mathcal{L}_nWT_X)(t_n,f) \quad (11)$$

where $\mathcal{L}_n$ conjugates the WT in (10) when n is even. For $N=2$, the HO-SCAL in (11) simplifies to the quadratic scalogram in (9).

Other members of the higher order affine class and their corresponding kernels are listed in Table 2. They include the multi-time/uni-frequency version of the higher order Choi-Williams distribution (HO-CWD) in [5], and the α form of the HO-WD (HO-αWD) that simplifies to the HO-WD for $\alpha=0$, and to the higher order Rihaczek distribution (HO-RD) for $\alpha=-1/N$.

2.2. Desirable properties of the higher order affine class

HO-TFRs in the higher order affine class defined in (7) always satisfy the scale covariance property in (4) for $N\in\mathbb{N}$ and the alternating sign time-shift covariance property in (5) for N even. Moreover, they satisfy additional desirable properties important for time-frequency analysis provided that their multi-dimensional

[2] By choosing the multi-time/uni-frequency form of the HO-WD, which is the dual of the uni-time/multi-frequency HO-WD in [4, 5], the resulting higher order affine class extends the affine class properties of scale covariance and time-shift covariance as opposed to extending the dual affine class properties of scale covariance and *frequency*-shift covariance. By the "dual" of a QTFR or HO-TFR, we mean interchanging signal terms x with spectral terms X, swapping temporal variables (e.g. time lag τ) with spectral variables (e.g. frequency lag ν) and interchanging a forward transform (e.g. FT) with its inverse transform, (e.g. inverse FT).

Normal form	Kernel	Higher order affine class HO-TFR, $A_X^N(t_1,\cdots,t_{N-1},f)$
A-I	ϕ_A^N	$\lvert f\rvert^{N-1}\int_\tau\int_{t'_1}\cdots\int_{t'_{N-1}}\phi_A^N(f(t_1-t'_1),\cdots,f(t_{N-1}-t'_{N-1}),f\tau)v_X^N(t'_1\cdots t'_{N-1},\tau)d\tau\prod_{n=1}^{N-1}dt'_n$
A-II	Φ_A^N	$\lvert f\rvert^{-1}\int_{f'}\int_{\nu_1}\cdots\int_{\nu_{N-1}}\Phi_A^N(-\frac{f'}{f},\frac{\nu_1}{f},\cdots,\frac{\nu_{N-1}}{f})V_X^N(f',\nu_1,\cdots,\nu_{N-1})df'\prod_{n=1}^{N-1}e^{j2\pi t_n\nu_n}d\nu_n$
A-III	ψ_A^N	$\lvert f\rvert^{N-2}\int_{f'}\int_{t'_1}\cdots\int_{t'_{N-1}}\psi_A^N(f(t_1-t'_1),\cdots,f(t_{N-1}-t'_{N-1}),-\frac{f'}{f})\text{WD}_X^N(t'_1,\cdots,t'_{N-1},f')df'\prod_{n=1}^{N-1}dt'_n$
A-IV	Ψ_A^N	$\int_\tau\int_{\nu_1}\cdots\int_{\nu_{N-1}}\Psi_A^N(f\tau,\frac{\nu_1}{f},\cdots,\frac{\nu_{N-1}}{f})\text{AF}_X^N(\tau,\nu_1,\cdots,\nu_{N-1})d\tau\prod_{n=1}^{N-1}e^{j2\pi t_n\nu_n}d\nu_n$
A-V	Γ_A^N	$\lvert f\rvert^{-1}\int_{f_0}\int_{f_1}\cdots\int_{f_{N-1}}\Gamma_A^N(\frac{f_0}{f},\frac{f_1}{f},\cdots,\frac{f_{N-1}}{f})\prod_{k=0}^{N-1}(\mathcal{L}_kX)(f_k)\,e^{j2\pi t_k(f_k-f_0)}df_k$

Table 1: *Five equivalent "normal forms" and their corresponding kernels for HO-TFRs, A_X, in the higher order affine class. In A-I, $v_X^N(t_1,\cdots,t_{N-1},\tau)=x^*(-\frac{\tau}{N}+\sum_{m=1}^{N-1}t_m)\prod_{n=1}^{N-1}(\mathcal{L}_nx)((-1)^{n+1}(t_n+\frac{\tau}{N}))$ is the Nth order temporal signal product. The Nth order spectral product, V_X^N, in A-II and the HO-WD, WD_X^N, in A-III are defined in (6). In A-IV, the multi-dimensional FT of the HO-WD [5] is the higher order ambiguity function: $AF_X^N(\tau,\nu_1,\cdots,\nu_{N-1})=\int_f\int_{t_1}\cdots\int_{t_{N-1}}WD_X^N(t_1,\cdots,t_{N-1},f)e^{j2\pi f\tau}df\prod_{n=1}^{N-1}e^{j2\pi t_n\nu_n}dt_n$.*

Affine HO-TFR	Normal form A-IV kernel $\Psi_A^N(\zeta,\beta_1,\cdots,\beta_{N-1})$	Properties
HO-WD	$\exp(-j2\pi\zeta)$	P1–P5
HO-RD	$\exp(-j2\pi\zeta)\prod_{i=1}^{N-1}\exp(-j2\pi\frac{\zeta\beta_i}{N})$	P1–P5
HO-αWD	$\exp(-j2\pi\zeta)\prod_{i=1}^{N-1}\exp(j2\pi\alpha\zeta\beta_i)$	P1–P5
HO-CWD	$\exp(-j2\pi\zeta)\prod_{i=1}^{N-1}\exp(-\frac{1}{\sigma}(\zeta\beta_i)^2)$	P1–P5
HO-SCAL	$f_r^{N-2}\text{AF}_H^{N*}(\frac{-c_1}{f_r},\cdots,\frac{-c_{N-1}}{f_r},-f_rb)$	P1, P2

Table 2: *Some affine HO-TFRs and their associated normal form A-IV kernel, $\Psi_A^N(\zeta,\beta_1,\cdots,\beta_{N-1})$, along with properties in Section 2.2 that each HO-TFR satisfies. In the scalogram kernel, $H(f)$ is the FT of the wavelet function $h(t)$ in (10). HO-TFR acronyms are defined in the text in Section 2.1*

kernel Ψ_A^N in normal form A-IV in Table 2 is properly constrained as shown below.

[$\boldsymbol{P}$-**1**] N**th order scale covariance**: The higher order affine class preserves scale changes on the analysis spectrum $X(f)$ as in (4). That is, if the frequency axis of $X(f)$ is dilated by the factor a, then so too is the frequency axis of A_X^N. The multi-dimensional time axes are compressed by the factor $1/a$.

[$\boldsymbol{P}$-**2**] **Alternating sign time-shift covariance**: Affine HO-TFRs (for N even) always preserve time shifts which alternate in sign along the multi-time axes as in (5). The alternating sign is due to the conjugation of every other signal term [10] using the conjugation operator $\mathcal{L}_n$ in (6). In particular, shifting the temporal signal by a constant τ translates the affine HO-TFR along its temporal axes by $\pm\tau$.

[$\boldsymbol{P}$-**3**] **Frequency-shift covariance**: Some affine HO-TFRs preserve constant frequency shifts. That is, $Y(f)=X(f-f_0)\Longrightarrow A_Y^N(t_1,\cdots,t_{N-1},f)=A_X^N(t_1,\cdots,t_{N-1},f-f_0)$, provided that $\Psi_A^N(\zeta,\beta_1,\cdots,\beta_{N-1})=e^{-j2\pi\zeta}\prod_{i=1}^{N-1}S(\zeta\beta_i)$. Here, $S(\eta)$ is a one-dimensional function. Affine HO-TFRs which satisfy $\boldsymbol{P}$-**3** are also members of the multi-time/uni-frequency version of Cohen's class [4, 5] (cf. section 3).

[$\boldsymbol{P}$-**4**] **Frequency marginal**: Integrating an affine HO-TFR along the multi-time axes equals the Nth order spectral moment: $\int_{t_1}\cdots\int_{t_{N-1}}A_X^N(t_1,\cdots,t_{N-1},f)\,dt_1\cdots dt_{N-1}=X^*(f)\prod_{i=1}^{N-1}(\mathcal{L}_iX)(f)$, provided $\Psi_A^N(\zeta,0,\cdots,0)=e^{-j2\pi\zeta}$.

[$\boldsymbol{P}$-**5**] **Frequency localization**: If the spectrum is perfectly concentrated along frequency $f=f_0$, then its affine HO-TFR will also be concentrated about the same frequency:

$X(f)=\delta(f-f_0)\Longrightarrow A_X^N(t_1,\cdots,t_{N-1},f)=\delta(f-f_0)$, provided $\Psi_A^N(\zeta,0,\cdots,0)=e^{-j2\pi\zeta}$.

In Table 2, we summarize examples of affine HO-TFRs, their corresponding kernels and the aforementioned properties that they each satisfy.

3. AFFINE–COHEN INTERSECTION OF HO-TFRS

In Section 2, we proposed the new affine class of HO-TFRs in (7) which satisfy the scale covariance property $\boldsymbol{P}$-**1** and the time-shift covariance property $\boldsymbol{P}$-**2**. In [5, 8], a dual higher order Cohen's class was proposed consisting of HO-TFRs, $C_X^N(t_1,\cdots,t_{N-1},f)$, which satisfy the time-shift covariance property $\boldsymbol{P}$-**2**, and the frequency-shift covariance property $\boldsymbol{P}$-**3**. These Cohen's class multi-time/uni-frequency HO-TFRs are expressed as:

$$C_X^N(t_1,\cdots,t_{N-1},f)=\int_\tau\int_{\nu_1}\cdots\int_{\nu_{N-1}}\text{AF}_X^N(\tau,\nu_1,\cdots,\nu_{N-1})\cdot\hat{\Psi}_C^N(\tau,\nu_1,\cdots,\nu_{N-1})e^{-j2\pi\tau f}d\tau\prod_{n=1}^{N-1}e^{j2\pi\nu_nt_n}d\nu_n \quad (12)$$

where $\hat{\Psi}_C^N$ is a kernel characterizing C_X^N, and AF_X^N is the higher order ambiguity function in Table 1. We propose the intersection subclass of the *new* higher order affine class and the higher order Cohen's class in (12). Thus, we want to group together HO-TFRs satisfying three covariance properties: scale-covariance $\boldsymbol{P}$-**1**, time-shift covariance $\boldsymbol{P}$-**2**, and frequency-shift covariance $\boldsymbol{P}$-**3**.

To obtain this new intersection, we find the affine HO-TFRs which satisfy $\boldsymbol{P}$-**3**. A sufficient condition for such affine class HO-TFRs is that they have kernels of the form, $\Psi_A^N(\zeta,\beta_1,\cdots,\beta_{N-1})=e^{-j2\pi\zeta}\prod_{i=1}^{N-1}S(\zeta\beta_i)$, where $S(\eta)$ is a one-dimensional, non-zero function. Similarly, we find the Cohen's class HO-TFRs which satisfy $\boldsymbol{P}$-**1**. Such Cohen's class HO-TFRs have kernels of the form, $\hat{\Psi}_C^N(\tau,\nu_1,\cdots,\nu_{N-1})=\prod_{i=1}^{N-1}S(\tau\nu_i)$, for the same one-dimensional function $S(\eta)$. Combining these two kernel constraints, we show that any HO-TFR in the higher order affine-Cohen intersection subclass, A-C_X^N, has an affine class kernel $\Psi_{\text{A-C}}^N$ that is related to its Cohen's class kernel $\hat{\Psi}_{\text{A-C}}^N$ as:

$$\Psi_{\text{A-C}}^N(\zeta,\beta_1,\cdots,\beta_{N-1})=e^{-j2\pi\zeta}\hat{\Psi}_{\text{A-C}}^N(\frac{\zeta}{f_r},f_r\beta_1,\cdots,f_r\beta_{N-1})=e^{-j2\pi\zeta}\prod_{i=1}^{N-1}S(\zeta\beta_n). \quad (13)$$

Substituting (13) into normal form A-IV, and also into normal form A-I using equation (8), members of this intersection subclass, A-C_X^N, can be expressed in terms of the one-dimensional function $S(\eta)$ or its inverse FT $s(\lambda)$, respectively as:

$$\text{A-C}_X^N(t_1,\cdots,t_{N-1},f) = \int_\tau\int_{\nu_1}\cdots\int_{\nu_{N-1}} \text{AF}_X^N(\tau,\nu_1,\cdots,\nu_{N-1}) e^{-j2\pi f\tau} \{\prod_{n=1}^{N-1} S(\tau\nu_n) e^{j2\pi t_n \nu_n} d\nu_n\} d\tau$$
$$= \int_\tau\int_{t_1'}\cdots\int_{t_{N-1}'} v_X^N(t_1-t_1',\cdots,t_{N-1}-t_{N-1}',\tau) \Big\{\prod_{n=1}^{N-1} \frac{1}{|\tau|} s(\frac{t_n'}{\tau}) dt_n'\Big\} e^{-j2\pi f\tau} d\tau. \quad (14)$$

Thus, (14) extends to higher order the quadratic affine-Cohen intersection subclass in [7]. Note that any HO-TFR in the intersection in (14) has a relatively simple structure that may facilitate analysis and expedite computation.

Figure 1 depicts the higher order affine-Cohen intersection. Some intersection members include the HO-WD with $S(\eta) = 1$, the HO-RD with $S(\eta) = e^{-j2\pi\frac{\eta}{N}}$, the HO-$\alpha$WD with $S(\eta) = e^{j2\pi\alpha\eta}$, and the HO-CWD with $S(\eta) = e^{-\eta^2/\sigma}$ (cf. Table 2). In contrast, the affine HO-SCAL in Table 2 does not satisfy ***P*-3** and is not a member of the higher order Cohen's class.

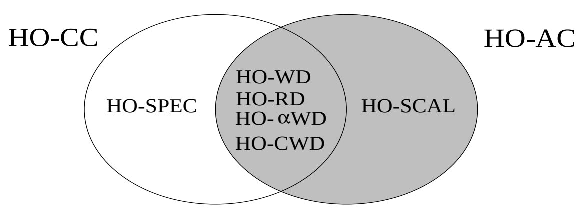

Figure 1. *The intersection of the higher order affine class (HO-AC) with the higher order Cohen's class (HO-CC) is depicted. Some intersection members include the HO-WD, HO-RD, HO-αWD, and the HO-CWD. The higher order spectrogram (HO-SPEC) [4] and the* new *HO-SCAL (higher order $|$wavelet transform$|^2$) are members of only the HO-CC and HO-AC, respectively. The shaded region corresponds to the* new *class proposed in this paper.*

4. CONCLUSION

In this paper, we proposed the *new* higher order affine class of HO-TFRs that preserve scale changes and time shifts in (4) and (5). It extends to higher order ($N > 2$) the second order ($N = 2$) affine class of scale covariant and time-shift covariant QTFRs. We derived five alternative "normal form" expressions of an affine HO-TFR in terms of multi-dimensional kernel functions. We discussed desirable properties and derived the corresponding kernel constraints necessary for an affine HO-TFR to satisfy these properties. We discussed important members of the *new* higher order affine class, including the *new* higher order scalogram. Finally, we provided a unifying framework for the intersection of the *new* higher order affine class with the previously developed higher order Cohen's class, by showing how their respective kernels are related. We also provided a simplified formulation for any member of the new higher order affine-Cohen intersection subclass in terms of a one-dimensional function.

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