OPTIMUM SUBBAND CODING OF CYCLOSTATIONARY SIGNALS

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ABSTRACT

We consider the optimal orthonormal subband coding of zero mean cyclostationary signals, with N-periodic second order statistics. A 2-channel uniform filter bank, with N-periodic analysis and synthesis filters, is used as the subband coder. A dynamic scheme involving N-periodic bit allocation is employed. An average variance condition is used to measure the output distortion. The conditions for maximizing the coding gain parallel those for the case when the signals are Wide Sense Stationary (WSS) and the analysis and synthesis filters and the bit allocation time invariant, in that the *blocked* subband signals must be decorrelated and the subband power spectral densities must obey an ordering. Some additional conditions on this ordering, over and above those required for the WSS case, are needed.

1. INTRODUCTION

We consider the optimum orthonormal subband coding of zero mean wide sense cyclostationary (WSCS) signals. A signal, $x(k)$ is WSCS with period N if for all k, l: $E[x(k)x(l)] = R_x(k, l) = R_x(k + N, l + N)$. A wide variety of man made signals encountered in communication, telemetry, radar and sonar systems, as well as several generated by nature [1], are WSCS.
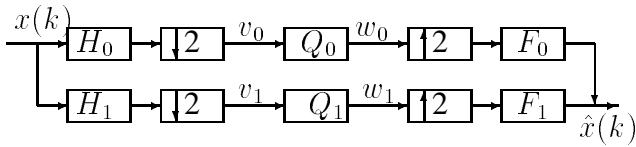

Figure 1: A two channel filter bank

The subband coder considered here is a 2-channel uniform maximally decimated filter bank depicted in fig. 1.

Here $x(k)$ is WSCS with period N and at time k, Q_i is a $b_i(k)$-bit quantizer. When $x(k)$ is Wide Sense Stationary (WSS) one selects the analysis filters $H_i(k, z^{-1})$ and the synthesis filters $F_i(k, z^{-1})$ to be linear time invariant (LTI). Since $x(k)$ here is WSCS with period N, we assume that these are *Linear Periodically Time Varying (LPTV) with period N*. A linear filter with impulse response $h(k, l)$ is called N-periodic, if for all k, l: $h(k, l) = h(k + N, l + N)$. The time index k in $H_i(k, z^{-1})$ and $F_i(k, z^{-1})$ recognizes their lack of time invariance. We assume that the filter bank is orthonormal, i.e. for all square summable inputs $x(k)$, the combined energy of the two subband signals $v_i(k)$ equals the energy in $x(k)$, and in the absence of the quantizers the filter bank output $\hat{x}(k)$ matches $x(k)$ for all $x(k)$. Under these conditions the subband signals are WSCS with period N. Accordingly the bit allocation scheme we will adopt is what we call *Periodically Dynamic Bit Allocation* (PDBA), where we choose each $b_i(k)$ to be N-periodic, i.e

$$b_i(k + N) = b_i(k). \tag{1}$$

Our goal is to select $b_i(k)$ and the filters $H_i(k, z^{-1})$ and $F_i(k, z^{-1})$ so that the *average* variance of $\hat{x}(k) - x(k)$ is the minimum, subject to orthonormality and a constant average transmission bit rate, i.e. for a fixed given b and all k

$$b = (b_1(k) + b_2(k))/2. \tag{2}$$

This consequently *maximizes the coding gain*. The optimization is performed on the basis of the second order statistics of $x(k)$. (Consult [2] on estimating the statistics of WSCS signals.)

Past work on the orthonormal subband coding of signals includes [3]- [5]. The work that most influences this paper is by Vaidyanathan, [6]. Assuming that $x(k)$ is WSS, zero mean, and using its second order statistics, [6] provides a complete solution to the problem of obtaining the *uniform* optimal M-channel *orthonormal* filter bank that maximizes the coding gain for this $x(k)$.

Since [6] influences our paper quite heavily, we briefly outline some of the ingredients of its analysis in Section

2. Section 3 gives preliminaries and reduces the optimum subband coding problem described above to a precise mathematical problem. Since LPTV systems can be converted by the *blocking* procedure to LTI systems, it may at first sight appear that the theory developed in [6] should extend very easily to the framework we plan to consider. *However, Section 3 demonstrates that the presence of the quantization devices underlying subbband coders changes the very nature of the optimization criterion, making the extension sought here nontrivial.* Section 4 gives the main results. Section 5 concludes. All proofs are omitted because of space constraints and can be found in [7] and in [8], a forthcoming journal version of this paper.

2. THE WSS CASE

In this section we recount the essentials of the result of [6], which considers the case where $x(k)$ is WSS, the analysis and synthesis filters are all LTI and the quantizer bits b_i are constant. Though the result of [6] applies to filter banks having arbitrary number of channels, we will consider only the 2-channel case. The quantizer noise model used by [6] is:

$$w_i(k) = v_i(k) + q_i(k) \tag{3}$$

where $q_i(k)$ is zero mean, independent from $v_i(k)$ and has variance

$$\sigma_{q_i}^2 = c2^{-2b_i}\sigma_{v_i}^2, \tag{4}$$

$\sigma_{v_i}^2$ is the variance of the subband signal v_i, and c is a constant determined by the signal distribution. Now the 2-channel filter bank of fig. 1 has the equivalent representation of fig. 2, with the 2×2 operators $E(z^{-1})$ and $R(z^{-1})$ LTI.
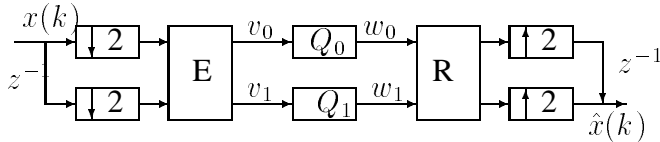

Figure 2: Polyphase Representation

Orthonormality reduces to the requirement that $R(z^{-1}) = E^{-1}(z^{-1})$ and for all ω

$$E'(e^{j\omega})E(e^{-j\omega}) = R'(e^{j\omega})R(e^{-j\omega}) = I. \tag{5}$$

Under these conditions given a set of $v_i(k)$, the mean square distortion at the output is minimized if for all i, j, $2^{-2b_0}\sigma_{v_0}^2 = 2^{-2b_1}\sigma_{v_2}^2$. In this case the quantity to be minimized becomes

$$J_{SBC} = \sigma_{v_0}^2\sigma_{v_1}^2. \tag{6}$$

Vaidyanathan, [6], provides necessary and sufficient conditions under which (6) is a minimum. The condition has two parts. First it is necessary that the subband signals

be mutually decorrelated: i.e. for all $i \neq j$, and k, l, $E[v_i(k)v_j(l)] = 0$. If one defines the subband signal vector $v(k) = [v_0(k), v_1(k)]'$ and its power spectral desity (PSD) matrix as $S_v(\omega)$, then this decorrelation entails that $S_v(\omega)$ be diagonal. This however doesn't by itself suffice for optimality. If one expresses

$$S_v(\omega) = \Lambda(\omega) = \text{diag}\{\lambda_0(\omega), \lambda_1(\omega)\} \tag{7}$$

then optimality is ensured if the $\lambda_i(\omega)$ obey a consistent magnitude ordering at each frequency, e.g.

$$\lambda_0(\omega) \geq \lambda_1(\omega) \text{ for all } \omega, \tag{8}$$

or the reverse. This is simply an energy compaction condition. Call the vector of inputs to $E(z^{-1})$, in fig. 2, $\bar{x}$ and its PSD matrix $S_{\bar{x}}(\omega)$. Then one can write

$$S_v(\omega) = E(e^{-j\omega})S_{\bar{x}}(\omega)E'(e^{j\omega}). \tag{9}$$

Since (5) holds, the $\lambda_i(\omega)$ are the eigenvalues of $S_{\bar{x}}(\omega)$.

3. PROBLEM FORMULATION

We now address the case where $x(k)$ is WSCS with period N, and $H_i(k, z^{-1})$ and $F_i(k, z^{-1})$ are N-periodic. In this Section we: (i) give a quantizer noise model; (ii) define a measure for the output distortion; (iii) give an optimum bit allocation scheme; and (iv) subject to optimum bit allocation, extract a precise mathematical optimization problem. Clearly, the subband signals are themselves WSCS with period N, i.e.

$$\sigma_{v_i}^2(k) = \sigma_{v_i}^2(k + N).$$

Extending [6], we will assume that the quantizers are modeled by additive zero mean noise sources, independent of the $v_i(k)$ with variances of the form

$$\sigma_{q_i}^2(k) = c2^{-2b_i(k)}\sigma_{v_i}^2(k). \tag{10}$$

Under (1) these are N-periodic. Even when the analysis and synthesis filters are time varying the polyphase representation depicted in fig. 2 still applies [9]. In this case the 2×2 operators $E(k, z^{-1})$ and $R(k, z^{-1})$ are N-periodic. Further the arrangement between the decimators and the quantizers can be replaced by the $2N \times 2N$ *blocked* LTI operator $\tilde{E}(z^{-1})$ depicted in fig. 3. Its input and output, the $2N \times 1$ vectors $\tilde{x}(k)$ and $\tilde{v}(k)$, are the respective blocked versions of the input and output to $E(k, z^{-1})$ in fig. 2. Notice the order in which the output samples of $v_i(k)$ appear in $\tilde{v}(k)$: i.e. the time indices of the appearances of $v_1(k)$ are in reverse order to those of $v_0(k)$.

$$\tilde{v}(k) = [v_0(Nk), v_0(Nk + 1), \cdots, v_0(Nk + N - 1),$$
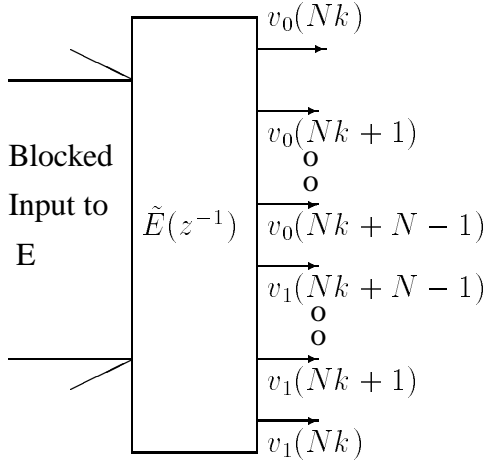$$v_1(Nk + N - 1), \cdots, v_1(Nk)]' \tag{11}$$

Figure 3: Blocked Analysis Bank

This arrangement simplifies the subsequent notation.

When $x(k)$ is WSCS with period N, both $\tilde{v}(k)$ and $\tilde{x}(k)$ are WSS with PSD matrices $S_{\tilde{v}}(\omega)$ and $S_{\tilde{x}}(\omega)$ respectively. Note that $S_{\tilde{x}}(\omega)$ can be constructed from the periodic autocorrelation, $R_x(k, l)$, and will be assumed to be available. Denote $\tilde{R}(z^{-1})$ to be the $2N \times 2N$ LTI operator representing the blocked version of the synthesis side of fig. 2.

Under orthonormality $\tilde{R}(z^{-1}) = \tilde{E}^{-1}(z^{-1})$ and

$$\tilde{E}'(e^{j\omega})\tilde{E}(e^{-j\omega}) = \tilde{R}'(e^{j\omega})\tilde{R}(e^{-j\omega}) = I. \qquad (12)$$

Clearly both $\hat{x}(k)$ and $\hat{q}(k) = \hat{x}(k) - x(k)$ are also WSCS with period N. Thus we propose to minimize the *average* variance of $\hat{q}(k)$, i.e.

$$\frac{1}{N}\sum_{i=0}^{N-1}\sigma_{\hat{q}}^2(i).$$

Because of (12) and (10), this in turn equals

$$\frac{c}{2N}\sum_{k=0}^{N-1}\left(2^{-2b_0(k)}\sigma_{v_0}^2(k) + 2^{-2b_1(k)}\sigma_{v_1}^2(k)\right). \qquad (13)$$

Since the arithmetic mean is lower bounded by the geometric mean with the bound achieved under equality of the elements, subject to (2), (13) is minimized if at each k

$$2^{-2b_0(k)}\sigma_{v_0}^2(k) = 2^{-2b_1(k)}\sigma_{v_1}^2(k). \qquad (14)$$

This is the *optimum bit allocation* scheme. Under (14) the minimization of (13) is equivalent to the minimization of

$$J_{SBC} = \sum_{j=0}^{N-1}\sqrt{\sigma_{v_0}^2(j)\sigma_{v_1}^2(j)}. \qquad (15)$$

In (15) the first output of $\tilde{E}(z^{-1})$ (fig. 3), is paired with the last the second with the second last, etc.. Hence we have the following precise mathematical problem.

Problem 1 *Consider the $2N \times 2N$ system $\tilde{E}(z^{-1})$ with WSS input vector $\tilde{x}(k)$ with positive definite Hermitian symmetric PSD matrix $S_{\tilde{x}}(\omega)$. Suppose $\tilde{v}(k)$ in (11) is the output of $\tilde{E}(z^{-1})$. Find $\tilde{E}(z^{-1})$ such that (15) is minimized subject to (12).*

The difference between the WSS case considered in [6] and the WSCS case, is evident. Whereas in [6] one minimizes the product of variances, (6), in this case it is the sum of square root of 2-products that must be minimized. The difference can be traced to the fact that while in the WSS case there is only one bit budget to be satisfied, in the N-periodic WSCS case *N bit budgets must be met* (2).

4. THE MAIN RESULT

In this Section we provide the solution to Problem 1. Though we will not prove the main result we will give a key intermediate Lemma that illustrates the approach taken in the proof. This Lemma employs the notion of majorization, [10] described below.

Definition 1 *Consider two sequence of numbers $h = \{h_i | 1 \le i \le n\}$ and $g = \{g_i | 1 \le i \le n\}$ with $h_i \ge h_{i+1}$ and $g_i \ge g_{i+1}$. Then we say that g majorizes h, i.e. $h \prec g$ if*

$$\sum_{i=1}^{l}h_i \le \sum_{i=1}^{l}g_i, \quad 1 \le l \le n \qquad (16)$$

with equality at $l = n$.

An important fact concerning majorization is the following, [10]:

Fact 1 *The diagonal elements of a positive definite Hermitian matrix are majorized by its eigenvalues.*

Since from (9) the eigenvalues of $S_v(\omega)$ are the same as those of $S_{\tilde{x}}(\omega)$, in the WSS case the normalized integral of the diagonal elements of all achievable $S_v(\omega)$, and hence all achievable subband variances, are majorized by the normalized integrals of $\lambda_i(\omega)$ in (8).

Also, if all elements of h and g are nonnegative, $h \prec g$ implies that, [10],

$$\prod_{i=1}^{n}h_i \ge \prod_{i=1}^{n}g_i. \qquad (17)$$

This is effectively a proof of [6]. In the WSCS case we are *not concerned with products of diagonal elements of the normalized integral of $S_{\tilde{v}}(\omega)$ but sum of square root of products*. Thus, (17) is of little value. Lemma 1 however, proves that a result comparable to (17) is possible.

Lemma 1 *Consider two sequences $\mathcal{A} = \{a_1, \cdots, a_{2N}\}$ and $\mathcal{B} = \{b_1, \cdots, b_{2N}\}$. Assume $a_i \geq a_{i+1} > 0$ and $b_i \geq b_{i+1} > 0$ and $\mathcal{A} \prec \mathcal{B}$. Consider a pair of disjoint partitions $\{i_1, i_2, \cdots, i_N\}$ and $\{l_1, l_2, \cdots, l_N\}$ of $\{1, 2, \cdots, 2N\}$. Then*

$$\sum_{n=1}^{N} \sqrt{a_{i_n} a_{l_n}} \geq \sum_{n=1}^{N} \sqrt{a_i a_{2N-i+1}}$$

$$\geq \sum_{n=1}^{N} \sqrt{b_i b_{2N-i+1}}.$$

Further equality holds in the second inequality iff $\mathcal{A} = \mathcal{B}$.

This result states two facts. First that among all possible sums of square root of products the minimum is attained by pairing the largest with the smallest, the second largest with the second smallest, etc.. Second, the sequence that majorizes the other yields a smaller value for this quantity. Now the main result:

Theorem 1 *Consider problem 1 and all quantities defined in its statement. Then optimality is attained only if the PSD matrix $S_{\tilde{v}}(\omega)$ is diagonal. Further suppose that,*

$$S_{\tilde{x}}(\omega) = \tilde{U}(e^{-j\omega})\tilde{\Lambda}(\omega)\tilde{U}'(e^{j\omega})$$

$\tilde{U}(e^{-j\omega})$ unitary at all ω, and

$$\tilde{\Lambda}(\omega) = diag\{\tilde{\lambda}_0(\omega), \cdots . \tilde{\lambda}_{2N-1}(\omega)\}$$

obeying at all ω

$$\tilde{\lambda}_i(\omega) \geq \tilde{\lambda}_{i+1}(\omega) > 0. \tag{18}$$

Then $\tilde{E}(z^{-1}) = \tilde{U}'(e^{j\omega})$ is one optimizing solution. In this case $S_{\tilde{v}}(\omega) = \tilde{\Lambda}(\omega)$.

Of course the $H_i(k, z^{-1})$ and $F_i(k, z^{-1})$ can be obtained by unblocking $\tilde{E}(z^{-1})$ and $\tilde{E}^{-1}(z^{-1})$. There are appealing parallels to the WSS case. The diagonal nature of $S_{\tilde{v}}(\omega)$ entails the complete mutual decorrelation of the blocked subband signals. Further, (18) parallels (8). *However, in addition the blocked subband signals must be paired in an appropriate way.* The largest energy signal must be paired with the smallest, the second largest with the second smallest etc. The solution given in Theorem 1 also reflects an energy compaction condition as all samples of the periodic variance of $v_0(k)$ dominate those of $v_1(k)$. Yet the situation now is more complicated. For example an arrangement in which λ_{2N-2} and λ_2 are the respective PSD's of $v_0(Nk)$ and $v_1(Nk)$, but λ_0 and λ_{2N-1} are the respective PSD's of $v_0(Nk+1)$ and $v_1(Nk+1)$ is permissible. Nevertheless, it is possible to show that the optimizing design can be effected using a class of periodic optimal compaction filters, see [8] for details.

5. CONCLUSION

We have given conditions for the optimal orthonormal subband coding of WSCS signals with period N, when the coder is a 2-channel uniform filter bank with N-periodic analysis and synthesis filters. A PDBA scheme involving N-periodic bit allocation is employed. An average variance condition is used to measure the output distortion. The optimality conditions parallel those for the WSS case in that the *blocked* subband signals must be decorrelated and the subband PSD's must obey an ordering. Unlike the WSS case some additional conditions on this ordering are required.

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