PERIODICALLY NONUNIFORM BANDPASS SAMPLING AS A TAPPED-DELAY-LINE FILTERING PROBLEM

Dan Scholnik and Jeffrey O. Coleman

scholnik@nrl.navy.mil | Naval Research Laboratory, Radar Division
jeffc@alum.mit.edu | Washington, DC 20375-5336

ABSTRACT

In this paper we consider systems for demodulation/modulation which use periodically nonuniform sampling (of arbitrary order) of the bandpass signal to circumvent the carrier-frequency restrictions of uniform sampling. The design of a particular tapped-delay-line (demodulation) or piecewise-constant-impulse-response (modulation) equivalent filter determines both the actual implementation filters and system performance. The tap spacing of the former and the transition times of the latter are periodically nonuniform. Following a characterization of the equivalent filter response, the special case of second-order sampling is examined for insight into the choice of sampling offset. A set of example designs demonstrates that, while nonuniform sampling permits carrier frequencies not allowed with uniform sampling, the resulting system performance is limited by the choice of carrier frequency.

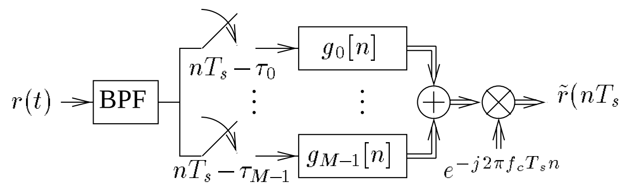
1. INTRODUCTION

When uniformly spaced complex-envelope samples are derived directly from an analog bandpass signal through uniform sampling, carrier-frequency choices are limited. Here periodically nonuniform sampling circumvents those restrictions in both the demodulator and modulator. After describing the systems, we consider the design of the equivalent filters that are key to their operation.

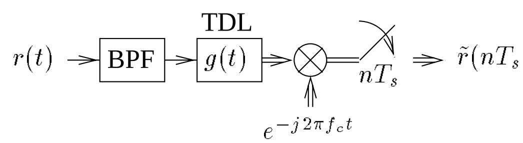
Figure 1(a) shows a demodulator system based on periodically nonuniform sampling. The incoming signal is bandpass filtered and fed to M samplers, each operating at a rate of $1/T_s$. Sampler m acquires samples at times $\mathbb{Z}T_s - \tau_m$, $\tau_m \in \mathbb{R}$, and $\cup_{m=0}^{M-1}(\mathbb{Z}T_s - \tau_m)$ constitutes the set of all required samples. In the equivalent, easier-to-analyze system of Fig. 1(b), the complex tapped-delay-line (TDL) filter impulse response $g(t)$ has support on $\cup_{m=0}^{M-1}(\mathbb{Z}T_s + \tau_m)$. The filters of Fig. 1(a) are discrete-time versions of the polyphase components of this single filter as illustrated in the first four lines of Fig. 2. Figure 3(a) shows the four signals in Fig. 1(b) (with the two filters considered as a unit). The filter combination recovers and shapes just the positive-frequency portion of the bandpass input signal prior to frequency shifting and sampling. (The aliasing is desired for the communication waveform shown, but in many applications the final sampling rate must be sufficient to avoid aliasing.)

In the modulation/reconstruction process, the real parts of the outputs of M discrete-time filters are nonuniformly interleaved for D/A conversion and bandpass filtering as shown in Fig. 4(a). The D/A receives samples from filter m at times $\mathbb{Z}T_s - \tau_m$ and

Scholnik's involvement in this work was supported by National Science Foundation grant MIP-9896034 to the University of Maryland Baltimore County, where he is a graduate student and Coleman is an Adjunct Assistant Professor of Computer Science and Electrical Engineering.



(a) Demodulation using periodically nonuniform sampling.



(b) Demodulation using uniform sampling and a TDL with nonuniformly spaced taps.

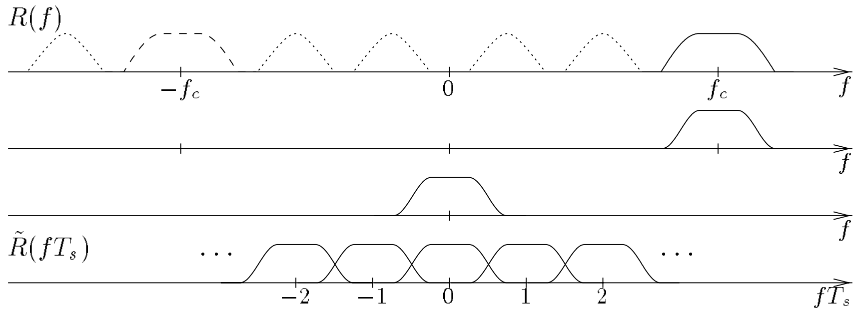
Figure 1: Demodulator system (a) is for implementation. Demodulator system (b) is for analysis and design.

outputs a stairstep with periodically nonuniform step widths. The mathematically equivalent system of Figure 4(b) features a single equivalent complex piecewise-constant-impulse-response filter related to the component filters as illustrated in Figure 2. Figure 3(b) shows representative signals. After the desired spectral component is shifted to the carrier frequency, the filters must together suppress the other components (with optional spectral shaping). The bandpass signal is then just the real part.

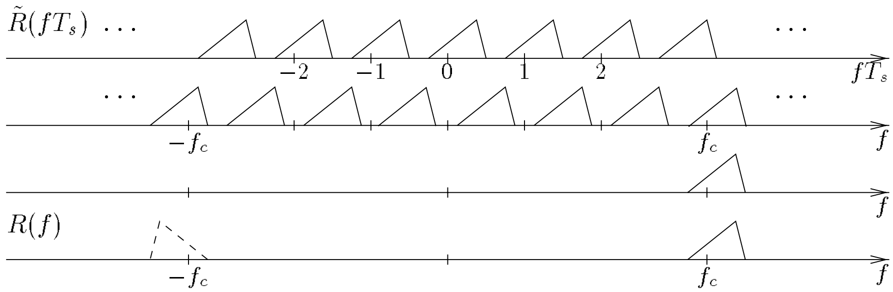
Bandpass sampling theory was first advanced for uniform and second-order cases [1], then specialized to quadrature sampling [2] (a special case of second-order sampling for I/Q recovery) and extended to higher-order periodically nonuniform sampling [3] and to multiple bands [4]. Systems using these ideas for demodulation almost always employ uniform and/or quadrature sampling [5–8]. The corresponding modulators [6, 9] are less often discussed.

2. DESIGNING THE NONUNIFORM FILTER

The frequency response of a uniformly spaced TDL filter cannot be independently specified at frequencies separated by integral multiples of the frequency-response period. What is the corresponding restriction for the (in general) aperiodic response associated with periodically nonuniformly spaced taps? It will be convenient to borrow the concept of an $\mathrm{aliasfree}(f_0)$ zone [4], the spectral support of a signal which can be sampled at a rate f_0 without aliasing. From a filter viewpoint, an $\mathrm{aliasfree}(f_0)$ zone represents a set of frequencies on which a periodic frequency response with period f_0 can be independently specified without conflict. Since $\mathcal{F}$ is $\mathrm{aliasfree}(f_0)$ if no two frequencies in $\mathcal{F}$ are equal modulo f_0, it follows that $\mathcal{F}_m \equiv \{f + m(f)f_0 : f \in \mathcal{F}\}$ is also $\mathrm{aliasfree}(f_0)$



(a) Example spectra for the demodulator of Fig. 1(b) operating on a communication waveform, including spectral shaping and symbol-rate sampling.



(b) Example signal spectra for the modulator of Fig. 4(b).

Figure 3: Spectral descriptions of the demodulation and modulation systems.

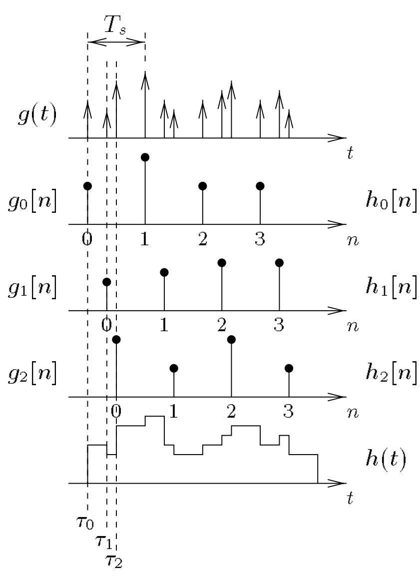
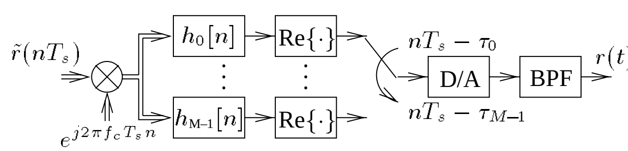


Figure 2: The modulator and demodulator equivalent filters are each related to (discrete-time representations of) their respective polyphase component filters as shown. (In general, modulator and demodulator filters would have different coefficients.)

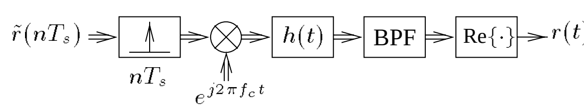
for any function $m : \mathcal{F} \to \mathbb{Z}$.

Suppose the impulse response of a TDL filter has support in $\mathcal{T} + \mathbb{Z}/f_0$, where $\mathcal{T}$ is a finite set. Then the frequency response of the filter can be written

$$G(f) = \sum_{\tau \in \mathcal{T}} G_\tau(f) e^{-j2\pi f \tau},$$



(a) Modulation using periodically nonuniform reconstruction



(b) Modulation using uniform reconstruction and a piecewise-constant filter with nonuniformly spaced transitions

Figure 4: Modulator system (a) is for implementation. Modulator system (b) is for analysis and design.

where polyphase component filter $G_\tau(f)$ has period f_0. For integer-valued function $m(f)$ then,

$$G(f + m(f) f_0) = \sum_{\tau \in \mathcal{T}} G_\tau(f) \gamma_\tau^{m(f)} e^{-j2\pi f \tau},$$

where $\gamma_\tau = e^{-j2\pi f_o \tau}$. If $\mathcal{T} = \{\tau_0 \ldots, \tau_{M-1}\}$, this can be written simultaneously for $m(f) = m_0(f), \ldots, m_{M-1}(f)$ as the matrix equation

$$\begin{pmatrix} G(f + m_0(f) f_0) \\ \vdots \\ G(f + m_{M-1}(f) f_0) \end{pmatrix} = \Gamma(f) \mathbf{D}(f) \begin{pmatrix} G_{\tau_0}(f) \\ \vdots \\ G_{\tau_{M-1}}(f) \end{pmatrix}, \tag{1}$$

with matrices $\Gamma(f)$, $\mathbf{D}(f)$ defined as

$$\Gamma(f) = \begin{pmatrix} \gamma_{\tau_0}^{m_0(f)} & \cdots & \gamma_{\tau_{M-1}}^{m_0(f)} \\ \vdots & & \vdots \\ \gamma_{\tau_0}^{m_{M-1}(f)} & \cdots & \gamma_{\tau_{M-1}}^{m_{M-1}(f)} \end{pmatrix},$$

$$\mathbf{D}(f) = \begin{pmatrix} e^{-j2\pi f\tau_0} & & \\ & \ddots & \\ & & e^{-j2\pi f\tau_{M-1}} \end{pmatrix}.$$

If f is restricted to $\mathrm{aliasfree}(f_0)$ zone $\mathcal{F}$, say $(-f_0/2, f_0/2)$, then (1) relates the TDL response $G(f)$ on M $\mathrm{aliasfree}(f_0)$ frequency-axis segments to the frequency responses on $\mathcal{F}$ of the polyphase component filters $\{G_\tau(f) : \tau \in \mathcal{T}\}$. Matrix $\mathbf{D}(f)$ is unitary and hence invertible, so if $\Gamma(f)$ is also invertible, then the M responses of the polyphase component filters can be determined from the M segment responses of TDL filter $G(f)$. If Γ is singular its range space is of dimension less than M, and some of those TDL segment responses can be determined from the others; independent specification of $G(f)$ on this family of segments is not possible. Further, for any f the ratio of the L_2 norm of the vector of segment responses to the L_2 norm of the vector of polyphase-component responses is bounded between the maximum and minimum singular values of $\Gamma(f)$. If these values differ widely—condition number $\chi(\Gamma)$, the ratio of the maximum to minimum singular value, provides a measure—extreme behavior may be required of the polyphase component filters in order to effect modest equivalent-filter behavior. One special case resulting in all singular values being identical is uniform sampling: $\tau_i = i/(Mf_0)$ with $m_i(f) = i + k$, for any fixed $k \in \mathbb{Z}$.

The analysis is similar for the piecewise-constant-impulse-response equivalent modulator filter, except that the matrix $\mathbf{D}(f)$ must incorporate the frequency responses of the D/A hold functions of the various widths. As a result, $\mathbf{D}(f)$ becomes singular at certain frequencies and so cannot be ignored.

Equation (1) is closely related to Eq. (11) of [4], which describes the condition for exact reconstruction of the sampled analog signal. Here the ideal reconstruction filters have been replaced by TDL filters, and the ideal composite response by a nonuniform TDL response, which is to be designed. The analysis above does not provide a filter-design method, but rather guides the selection of sampling phases $\tau_1, \ldots, \tau_M$ so that FIR polyphase component filters of modest lengths will suffice. If the polyphase component filters are FIR, the frequency responses of the equivalent filters are, at any particular frequency, linear in the filter coefficients, so the filters can be designed with linear programming [10], semidefinite programming [11], or eigenfilter [12] and generalized-eigenfilter methods [13], for example.

3. A SPECIAL CASE: SECOND-ORDER SAMPLING

Some insight can be gained from analysis of second-order sampling, the simplest type of nonuniform sampling. Without loss of generality, we can assume that $\tau_0 = 0$, $\tau_1 = \tau$, $m_0(f) = 0$, and $m_1(f) = m(f)$ for f in some $\mathrm{aliasfree}(f_0)$ zone $\mathcal{F}$. This results in $\Gamma(f) = \begin{bmatrix} 1 & 1 \\ 1 & \gamma^{m(f)} \end{bmatrix}$ with $\gamma = e^{-j2\pi f_0 \tau}$. The condition number $\chi(\Gamma)$ is then a function of offset τ and integer $m(f)$, which defines the relationship between the two spectral regions ($\mathcal{F}$ and $\mathcal{F}_m$) to be designed. Figure 5 is a plot of the inverse condition

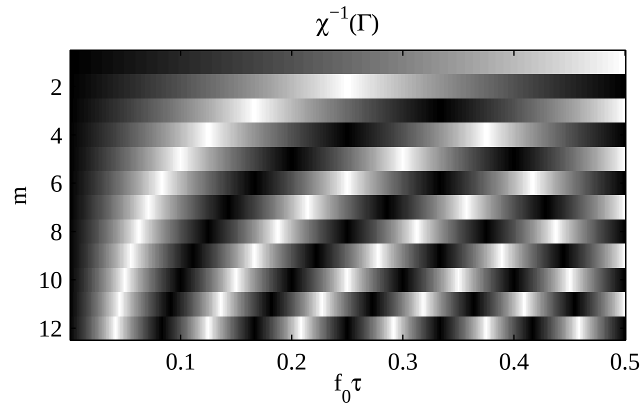


Figure 5: Inverse condition number of the matrix Γ vs. sampling offset τ and index m for the case of second-order sampling. White indicates a value of 1.

number vs. τ (normalized using f_0) and m, with white indicating the ideal case of unity and black representing a value of zero (meaning Γ is singular). Thus, light regions represent good combinations of time offset τ and relative frequency offset m.

We can use Fig. 5 to visualize some previous bandpass sampling theory results. Consider a real bandpass signal $X(f)$ with support in $(-f_c - f_0/2, -f_c + f_0/2) \cup (f_c - f_0/2, f_c + f_0/2)$, choosing $\mathcal{F}$ as the first term of the union, and choosing $m(f)$ such that $\mathcal{F}_m$ is equal to the second term. A value of $f_0\tau = 0.5$ represents uniform sampling, which is ideal for odd m but results in a singular Γ for even m, corresponding to the integer and half-integer band-position cases of [3]. For a bandpass signal with carrier frequency f_c a common sampling scheme is *quadrature sampling* [2], defined by $\tau \in \frac{2\mathbb{Z}+1}{4f_c}$. For $f_c = mf_0/2$, this corresponds to the white peaks of the figure and thus to a unity condition number of Γ. When f_c is not a half-integer multiple of f_0 the integer function $m(f)$ becomes piecewise constant, taking on the values of the two integers nearest to $2f_c/f_0$. In this case no value of τ can make $\chi(\Gamma) = 1$ for all f, and the quadrature sampling choice represents a compromise of sorts. This suggests that even with an arbitrary sampling offset, a demodulator can achieve better performance when the carrier frequency and sampling rate are half-integer related. An example design will help to illustrate.

Consider a second-order bandpass sampling system with $f_c = 1.7f_0$, where the TDL filter will be designed to suppress frequencies in the interval $[-2.1f_0, -1.3f_0] \subset \mathcal{F} = (-2.2f_0, -1.2f_0)$ and pass frequencies in $[1.3f_0, 2.1f_0] \subset \mathcal{F}_m = (1.2f_0, 2.2f_0)$ where $m(f)$ is as shown in Fig. 6 along with a potential input signal spectrum $R(f)$. The discontinuity in $m(f)$ causes a similar discontinuity in the "ideal" frequency responses of the two uniform TDL filters. (A similar result was found in [4] for analog reconstruction filters.) Although the goal is not to directly approximate each of the ideal responses with TDL filters (but rather to directly design $G(f)$, the single equivalent response), it is expected that this discontinuity will degrade performance. For comparison, we consider a similar system with $f_c = 1.5f_0$, where $m(f) = 3$, continuous on the interval of interest $\mathcal{F} = (-2f_0, -f_0)$. Choosing $\tau = 1/4f_c$ for both cases, linear programming was used to design the composite TDL responses as the sum of two offset 14-tap FIR filters, with the overall response constrained to have linear phase for simplicity. The peak stopband level was set to -60 dB and the passband ripple was minimized. The top two plots of Fig. 7 show the resulting responses and passband details. The second filter, corresponding to $f_c = 1.5f_0$ (and a constant $m(f)$) has less

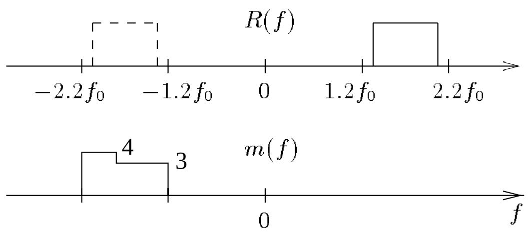


Figure 6: Example bandpass signal spectrum and corresponding function $m(f)$.

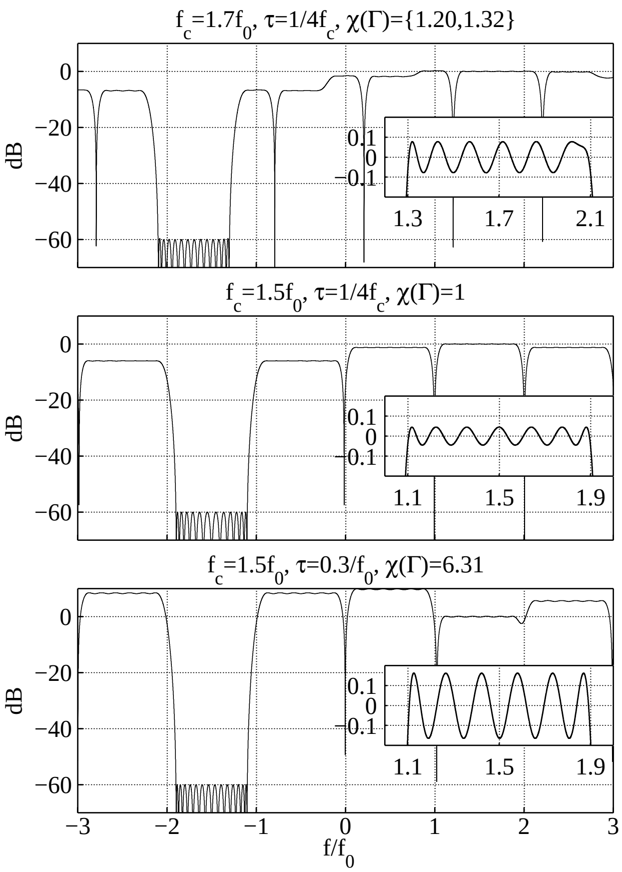


Figure 7: Frequency responses of the example nonuniformly-spaced TDL filters.

passband ripple than the top filter.

Even when $m(f)$ is constant on the interval of interest, the condition number of the matrix Γ has a strong effect on the performance of the nonuniform TDL filter. Consider again a second-order bandpass sampling system with $f_c = 1.5f_0$, but now with $\tau = 1/4f_c$ and $\tau = 0.3/f_0$. In the first case, the condition number is 1, but in the second case it is > 6. The second and third plots of Fig. 7 show the two responses. Both the passband ripple and the out-of-band filter response are larger for the high-condition-number case. This third response is in fact worse than the first case, where $m(f)$ was discontinuous, and $\chi(\Gamma) = \begin{cases} 1.2, & m = 3 \\ 1.3, & m = 4 \end{cases}$.

4. SUMMARY

The analysis of a certain class of periodically nonuniform sampling/reconstruction systems was presented as a filter design problem, where a single equivalent TDL filter response controls the spectral shaping of the sampled function. Second-order sampling was chosen for its simplicity to illustrate the concepts. It was seen that even when arbitrary sampling offsets are allowed, the relationship between the sampling rate and the spectral location of the bandpass signal can limit system performance.

5. REFERENCES

[1] A. Kohlenberg, "Exact interpolation of band-limited functions," *J. Appl. Phys.*, vol. 24, no. 12, Dec. 1953.

[2] O. D. Grace and S. P. Pitt, "Sampling and interpolation of bandlimited signals by quadrature methods," *J. Acoust. Soc. Amer.*, vol. 48, no. 6, 1970.

[3] Alan J. Coulson, "A generalization of nonuniform bandpass sampling," *IEEE Trans. Signal Processing*, vol. 43, no. 3, pp. 694–704, Mar. 1995.

[4] Yuan-Pei Lin and P. P. Vaidyanathan, "Periodically nonuniform sampling of bandpass signals," *IEEE Trans. Signal Processing II: Analog and Digital Signal Processing*, vol. 45, no. 3, Mar. 1998.

[5] W. M. Waters and B. R. Jarrett, "Bandpass signal sampling and coherent detection," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-18, no. 4, pp. 731–6, Nov. 1982.

[6] D. P. Scholnik and J. O. Coleman, "Simple, exact models of sample-interleaving demodulators/modulators for quadrature bandpass sampling/reconstruction," in *Proc. Conf. Information Science and Systems*, Baltimore, MD, Mar. 1997.

[7] D. P. Scholnik and J. O. Coleman, "Integrated I-Q demodulation, matched filtering, and symbol-rate sampling using minimum-rate IF sampling," in *Proc. 1997 Symp. Wireless Personal Communication*, Blacksburg, VA, June 1997.

[8] W. Rosenkranz, "Quadrature sampling of FM-bandpass signals—implementation and error analysis," in *Proc. of the International Conference Digital Signal Processing 87*, 1987, pp. 377–81.

[9] Henry Samueli and Bennett C. Wong, "A VLSI architecture for a high-speed all-digital quadrature modulator and demodulator for digital radio applications," *IEEE J. Select. Areas in Commun.*, vol. 8, no. 8, pp. 1512–1519, Oct. 1990.

[10] J. O. Coleman, "Choosing nonuniform tap spacings for a tapped-delay-line filter," *IEEE Trans. Circuits and Systems II*, Apr. 1996.

[11] S.-P. Wu, S. Boyd, and L. Vandenberghe, "FIR filter design via semidefinite programming and spectral factorization," in *Proc. 1996 IEEE Conf. Decision and Control (CDC '96)*, Dec. 1996, Kobe, Japan.

[12] P . P . Vaidyanathan and T. Q. Nguyen, "Eigenfilters: A new approach to least-squares FIR filter design and applications including Nyquist filters," *IEEE Trans. Circuits and Systems*, vol. CAS-34, no. 1, pp. 11–23, Jan. 1987.

[13] J. O. Coleman, "Embedded FIR generalized eigenfilters using test inputs," in *Proc. 1998 Int'l Conf. Acoustics, Speech, and Signal Processing (ICASSP '98)*, May 1998.