

SAMPLING ON UNIONS OF NON-COMMENSURATE LATTICES VIA COMPLEX INTERPOLATION THEORY

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ABSTRACT

Solutions to the analytic Bezout equation associated with certain multichannel deconvolution problems are interpolation problems on unions of non-commensurate lattices. These solutions provide insight into how one can develop general sampling schemes on properly chosen non-commensurate lattices. We will give specific examples of non-commensurate lattices, and use a generalization of B. Ya. Levin's sine-type functions to develop interpolating formulae on these lattices.

1. INTRODUCTION

¹Linear, translation invariant systems (e.g., sensors, linear filters) are modeled by the convolution equation $s = f * \mu$, where f is the input signal, μ is the system impulse response function (or, more generally, impulse response distribution), and s is the output signal. We refer to μ as a convolver. In many applications, the output s is an inadequate approximation of f , which motivates solving the convolution equation for f , i.e., deconvolving f from μ . If the function μ is *realizable*, i.e., time-limited (compactly supported) and non-singular, we have shown that this deconvolution problem is ill-posed in the sense of Hadamard (see [7]). A theory of solving such equations has been developed (see [1]–[7]). It circumvents ill-posedness by using a multichannel system. If we overdetermine the signal f by using a *system* of convolution equations, $s_i = f * \mu_i$, $i = 1, \dots, n$, the problem of solving for f is well-posed if the set of convolvers $\{\mu_i\}$ satisfies the condition of being what we call *strongly coprime*. This condition, among other things, gives that the Fourier-Laplace transforms $\widehat{\mu}_i$ have no common zeros. Thus, in

a strongly coprime system, no signal information is lost. Additionally, it guarantees the existence of compactly supported distributions (deconvolvers) ν_i , $i = 1, \dots, n$ such that $\widehat{\mu}_1 \cdot \widehat{\nu}_1 + \dots + \widehat{\mu}_n \cdot \widehat{\nu}_n = 1$. Transforming, we get $\mu_1 * \nu_1 + \dots + \mu_n * \nu_n = \delta$, which in turn gives $s_1 * \nu_1 + \dots + s_n * \nu_n = f$. The deconvolvers are inverse transforms of solutions to the *analytic Bezout equation*, i.e., for given holomorphic f_i and ϕ satisfying certain growth conditions, holomorphic g_i satisfying the same growth conditions such that $f_1 \cdot g_1 + \dots + f_n \cdot g_n = \phi$. For our purposes, we want growth conditions given by the Paley-Wiener-Schwartz Theorem and $\phi = \phi_\lambda$, with $\phi_\lambda \rightarrow 1$ as $\lambda \rightarrow \infty$ (ϕ_λ is the transform of an approximate identity). This theory is developed in detail in [6, 7].

The multichannel theory of deconvolution is intimately tied to the theory of sampling. This connection is not surprising. Basic sampling theory allows us to apply deconvolution to sampled band-limited signals, performing a deconvolution and a signal reconstruction in one processing step (see [7]). Moreover, the heart of the multichannel theory involves solving an interpolation problem, reconstructing (generalized) functions (the deconvolvers) in a space of restricted growth ($\widehat{\mathcal{E}}'$) from discrete data (their values on the zero sets of the convolvers). This gives solutions to the Bezout equation (see [6, 7]). This development utilizes the zero sets of the $\widehat{\mu}_i$ as different sampling rates.

The purpose of this note is to show how this connection between sampling and multichannel deconvolution naturally leads to sampling schemes on properly chosen non-commensurate lattices. We will then give specific examples of non-commensurate sampling lattices, and use a generalization of B. Ya. Levin's sine-type functions to develop interpolating formulae on these lattices. We close by presenting simulations of these results.

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2. SAMPLING AND THE BEZOUT EQUATION

A key step in the development of multichannel systems is the solution of the Bezout equation $\widehat{\mu}_1 \cdot \widehat{\nu}_1 + \dots + \widehat{\mu}_n \cdot \widehat{\nu}_n = \phi$, where $\phi = \phi_\lambda$, with $\phi_\lambda \rightarrow 1$ as $\lambda \rightarrow \infty$. This is, given the data (values of the $\widehat{\mu}_i$, especially the location of the zeros), a sampling problem. We can easily write down a solution to Bezout at those zeros. The problem now becomes one of interpolation. We need to interpolate the $\widehat{\nu}_i$ between these zeros so that they satisfy the necessary growth conditions. There are currently two approaches used to solve these problems – complex methods (Jacobi interpolation formulae and Cauchy residue theory) or real methods (sampling theory).

The development of the deconvolvers ν_i via the Jacobi interpolation formulae and Cauchy residue theory is now a well-developed tool, given the theoretical base established by Berenstein, Gay, Taylor, Yger, et al. ([1]–[6]). It has a flexibility in the one-variable case that allows for its use in not only general deconvolution problems, but also in the development of filters, etc. The key to this is the flexibility of the Cauchy residue calculus in one-variable. As is well-known, the story in several variables is different. Berenstein, Gay, Taylor, Yger, et al. have given us working formulae for specific situations. There are no general formulae in several variables for the computation of the needed residues.

Sampling lies juxtaposition these methods. It does not have the tremendous flexibility of the complex methods in one variable, but it also does not carry with it comparable computational difficulties in several variables ([6, 7, 10]). We will later use these techniques to develop non-commensurate sampling lattices. In this section, we cite two concrete examples in which sampling theory was used to solve the Bezout equation, which in turn gave us the deconvolvers.

Let $t \in \mathbf{R}$, p be a prime number, and let $\mu_1(t) = \chi_{[-1,1]}(t)$, $\mu_2(t) = \chi_{[-\sqrt{p},\sqrt{p}]}(t)$ model the impulse response of the channels of a two-channel system. Then $\widehat{\mu}_1(\zeta) = \frac{\sin(2\pi\zeta)}{\pi\zeta}$, $\widehat{\mu}_2(\zeta) = \frac{\sin(2\pi\sqrt{p}\zeta)}{\pi\zeta}$. Let $\mathcal{Z}_1 = \{\frac{\pm k}{2}\}$, $\mathcal{Z}_2 = \{\frac{\pm k}{2\sqrt{p}}\}$, for $k \in \mathbf{N}$ denote the zero sets of $\widehat{\mu}_1(\zeta)$, $\widehat{\mu}_2(\zeta)$, respectively. An examination of the Fourier-Laplace transforms $\widehat{\mu}_i(\zeta)$, $i = 1, 2$, gives that $\{\mu_i\}$ is strongly coprime (see [3]). We choose an arbitrarily close approximation ψ of the Dirac δ based on certain criteria, i.e., ψ in C^4 with support in $-(1 + \sqrt{p}), (1 + \sqrt{p})$. Then $|\widehat{\psi}(z)| \leq \frac{d}{(1+|z|)^4}$ for $z \in \mathcal{Z}_1 \cup \mathcal{Z}_2$. The smoothness and the size of the support of ψ guarantee that the deconvolvers are compactly supported.

We use Shannon sampling to create the $\nu_{i,\psi}(t)$. To give our formulae for the approximate deconvolvers $\nu_{1,\psi}$, $\nu_{2,\psi}$, we use the Jacobi interpolation formula to get a solution to the modified Bezout equation. We then show that these deconvolvers satisfy certain PW growth estimates. This in turn allows us to apply sampling. We use the zero sets of the transforms of the convolvers to give us the sampling rates. More general modifications of this construction do not use Jacobi interpolation (see [10]).

Theorem 1 ([6, 7]) *Let $\mu_1(t) = \chi_{[-1,1]}(t)$, $\mu_2(t) = \chi_{[-\sqrt{p},\sqrt{p}]}(t)$, for p prime. Let $f \in L^2(\mathbf{R})$ and let ψ be an even C^4 function with support in $-(1 + \sqrt{p}), (1 + \sqrt{p})$ such that $\psi \geq 0$ and $\int_{-\infty}^{\infty} \psi(t) dt = 1$. The deconvolvers $\nu_{i,\psi}$ such that*

$$f * \psi = (f * \mu_1) * \nu_{1,\psi} + (f * \mu_2) * \nu_{2,\psi}$$

are given by the formulae

$$\begin{aligned} \nu_{1,\psi}(t) &= \left(\frac{1}{2\sqrt{p}} \sum_{j \neq 0} (-1)^{j+1} \frac{\widehat{\psi}(j/2\sqrt{p})}{\widehat{\mu}_1(j/2\sqrt{p})} + \right. \\ &\quad \left. \frac{1}{2\sqrt{p}} \sum_{n \neq 0} \frac{\widehat{\psi}(n/2\sqrt{p})}{\widehat{\mu}_1(n/2\sqrt{p})} e^{\pi i(n/\sqrt{p})t} \right) \times \chi_{[-\sqrt{p},\sqrt{p}]}, \\ \nu_{2,\psi}(t) &= \left(\frac{1}{2} \sum_{j \neq 0} (-1)^{j+1} \frac{\widehat{\psi}(j/2)}{\widehat{\mu}_2(j/2)} + \right. \\ &\quad \left. \frac{1}{2} \sum_{n \neq 0} \frac{\widehat{\psi}(n/2)}{\widehat{\mu}_2(n/2)} e^{\pi i n t} \right) \times \chi_{[-1,1]}. \end{aligned}$$

We can also use modulation to create a strongly coprime system. This new technique for creating these systems allows for a greater flexibility in the development of actual systems. The system is created by making two identical copies of a given sensor, splitting the signal into two the two separate channels, and appropriate modulation of both the input and output of one of the channels. These two outputs are then convolved with the appropriate deconvolving filters and added, resulting in the reconstruction of the input signal.

Let μ_1 model the impulse response of a given system. Let $E_{\zeta_0} = e^{2\pi i \zeta_0 t}$. We will refer to E_{ζ_0} as a modulating function in the time domain. Multiplication by E_{ζ_0} is usually called “Quadrature Amplitude Shift Keying” or “Quadrature Amplitude Modulation.” “Quadrature” refers to the fact that the real and imaginary parts of the modulating function are $\frac{\pi}{2}$ out of phase with each other.

Lemma 1 ([6]) *If $f \in L^2$ and μ is realizable, then*

$$f * E_{\zeta_0} \mu = E_{\zeta_0} [E_{-\zeta_0} f * \mu]. \quad (1)$$

We apply Lemma 1 to create a multichannel system for a system with impulse response $\mu_1(t) = \chi_{[-1,1]}(t)$. The Fourier-Laplace transform of $\mu_1(t)$ is $\widehat{\mu}_1(\zeta) = \frac{\sin(2\pi\zeta)}{\pi\zeta}$. This has zeros $\mathcal{Z}_1 = \{\pm \frac{k}{2} : k \in \mathbf{N}\}$. Let $\mu_2(t) = E_{\frac{1}{4}}\mu_1 = e^{\frac{\pi}{2}it}\mu_1(t)$. Then $\widehat{\mu}_2(\zeta) = \frac{\sin(2\pi(\zeta - \frac{1}{4}))}{\pi(\zeta - \frac{1}{4})}$ and $\mathcal{Z}_2 = \{\frac{1}{4} \pm \frac{k}{2}\}^2$.

Theorem 2 ([6]) *The functions $\mu_1(t)$, $\mu_2(t)$ form a strongly coprime pair of convolvers.*

We construct deconvolvers $\nu_{i,\psi}$ such that for an approximate identity function ψ , we have $\mu_1 * \nu_{1,\psi} + \mu_2 * \nu_{2,\psi} = \psi$. We note that for $n \in \mathbf{Z}$, $\mathcal{Z}_1 \cup \mathcal{Z}_2 = \{\frac{n}{4}\} \setminus \{0, \frac{1}{4}\}$. Now, if we assume that the auxiliary function ψ has support $\subset (-2, 2)$, a construction of ψ via Shannon sampling in frequency is possible. The Nyquist rate is $\frac{1}{4}$ – exactly the same rate as $\mathcal{Z}_1 \cup \mathcal{Z}_2$. These observations again led us to use sampling to solve the Bezout equation, which, in turn, yielded the following.

Theorem 3 ([6]) *Let $f \in L^2(\mathbf{R})$, and let ψ be an even C^4 function with support in $(-2, 2)$ such that $\psi \geq 0$ and $\int_{-\infty}^{\infty} \psi(t) dt = 1$. Given*

$$\mu_1(t) = \chi_{[-1,1]}(t), \quad \mu_2(t) = e^{\frac{\pi}{2}it}\chi_{[-1,1]}(t),$$

the deconvolvers $\nu_{i,\psi}$ such that

$$f * \psi = (f * \mu_1) * \nu_{1,\psi} + (f * \mu_2) * \nu_{2,\psi}$$

are given by the formulae

$$\begin{aligned} \nu_{1,\psi}(t) &= \frac{1}{2} \left[K_1 + \sum_{n \neq 0} \frac{\widehat{\psi}((1/4) + (n/2))}{\widehat{\mu}_1((1/4) + (n/2))} e^{\pi int} \right] \\ &\quad \times e^{\frac{\pi}{2}it} \chi_{[-1,1]}(t), \end{aligned}$$

$$\nu_{2,\psi}(t) = \frac{1}{2} \left[K_2 + \sum_{n \neq 0} \frac{\widehat{\psi}(n/2)}{\widehat{\mu}_2(n/2)} e^{\pi int} \right] \times \chi_{[-1,1]}(t),$$

where

$$K_1 = \sum_{n \neq 0} (-1)^{n+1} \frac{\widehat{\psi}((1/4) + (n/2))}{\widehat{\mu}_1((1/4) + (n/2))},$$

$$K_2 = \sum_{n \neq 0} (-1)^{n+1} \frac{\widehat{\psi}(n/2)}{\widehat{\mu}_2(n/2)}.$$

*The function $f * \psi$ is an arbitrarily close approximation of f which converges to f in the sense of distributions as $\text{supp}(\psi) \rightarrow \{0\}$.*

²Not to be confused with the zero set of Theorem 1.

Remark : Thus, given a fixed system, it is possible to modify the system via some easily performed transform to create a strongly coprime system, and consequently recover the complete input function. This modulation technique works for creating strongly coprime systems for B-spline systems. In a similar vein, given $\chi_{[-1,1] \times [-1,1]}$, a strongly coprime system is created by rotating the square by $\frac{\pi}{4}$ and $\frac{\pi}{3}$ (see [4]).

3. SAMPLING ON NON-COMMENSURATE LATTICES

Again, let $t \in \mathbf{R}$, p be a prime number, and let $\mu_1(t) = \chi_{[-1,1]}(t)$, $\mu_2(t) = \chi_{[-\sqrt{p}, \sqrt{p}]}(t)$ model the impulse response of the channels of a two-channel system. Then $\widehat{\mu}_1(\zeta) = \frac{\sin(2\pi\zeta)}{\pi\zeta}$, $\widehat{\mu}_2(\zeta) = \frac{\sin(2\pi\sqrt{p}\zeta)}{\pi\zeta}$. Let $\mathcal{Z}_1 = \{\frac{\pm k}{2}\}$, $\mathcal{Z}_2 = \{\frac{\pm k}{2\sqrt{p}}\}$, for $k \in \mathbf{N}$ denote the zero sets of $\widehat{\mu}_1(\zeta)$, $\widehat{\mu}_2(\zeta)$, respectively. An examination of the Fourier-Laplace transforms $\widehat{\mu}_i(\zeta)$, $i = 1, 2$, gives that $\{\mu_i\}$ is strongly coprime.

Now let

$$\Gamma_1 = \left\{ \frac{\pm k}{2} \right\}, \quad \Gamma_2 = \left\{ \frac{\pm k}{2\sqrt{p}} \right\},$$

for $k \in \mathbf{N}$, and let

$$\Gamma = \Gamma_1 \cup \Gamma_2.$$

Note that the information contained in the original signal is reconstructed by creating deconvolvers defined initially on Γ . We can show the following.

Theorem 4 *Let p be a prime, and let f be a $(1 + \sqrt{p})$ -band-limited function. Then f is uniquely encoded on Γ .*

Note, for a $(1 + \sqrt{p})$ -band-limited function, the Nyquist rate is $1/(2(1 + \sqrt{p}))$. However, our individual sampling rates are $1/2$ and $1/(2\sqrt{p})$. Both these rates (and their average) are below Nyquist.

The reconstruction of f from this lattice is achieved by using complex interpolation theory. These techniques go back to basic Lagrange interpolation, and were developed for entire functions by various mathematicians, most notably B. Ya. Levin [8].

Theorem 5 *Let p be a prime, and let f be a $(1 + \sqrt{p})$ -band-limited function. Let $\Gamma_1 = \{\frac{\pm k}{2}\}$, $\Gamma_2 = \{\frac{\pm k}{2\sqrt{p}}\}$, for $k \in \mathbf{N}$, and let $\Gamma = \Gamma_1 \cup \Gamma_2$. Then f can be reconstructed from its values on Γ by the formula*

$$f(t) \approx \sum_{\lambda_k \in \Gamma} f(\lambda_k) \frac{S(t)}{S'(\lambda_k)(t - \lambda_k)} + f'(0) \frac{S(t)}{S'(0)(t)}, \quad (2)$$

where

$$S(t) = \sin(2\pi t) \cdot \sin(2\sqrt{p}\pi t). \quad (3)$$

Figures 1 – 3 give a simulation of this result.

The result also generalizes. We can create sampling sets on $k + 1$ lattices using 1 and the first k primes. We can also create these as unions of regular lattices in higher dimensions.

We close by pointing out two items. First, the sampling grid is rigid. Perturbation of the grid results in a loss of information. Second, that because sampling points in Γ can get arbitrarily close together, the interpolating formula does not converge in norm. The interpolating function follows the original function along exactly, except for a very subtle “ripple” at those points where the sampling points get close together. We are currently exploring ways to stabilize the construction. The first involves continuously backing off the bandwidth. This works, but we need to set up exact bounds. The second approach involves rewriting Γ in terms of the separated and “close” points. At the close points, we can rewrite the formulae in terms of f' .

4. REFERENCES

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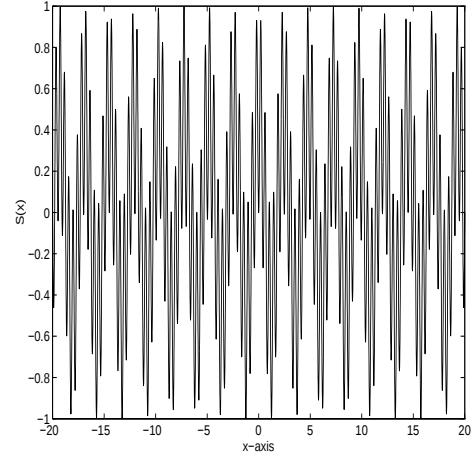


Figure 1: The component S of the interpolating function.

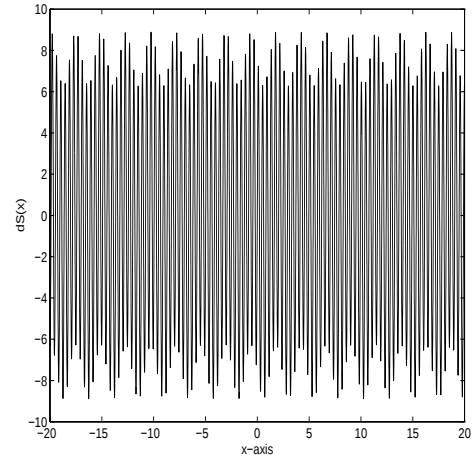


Figure 2: The component S' of the interpolating function.

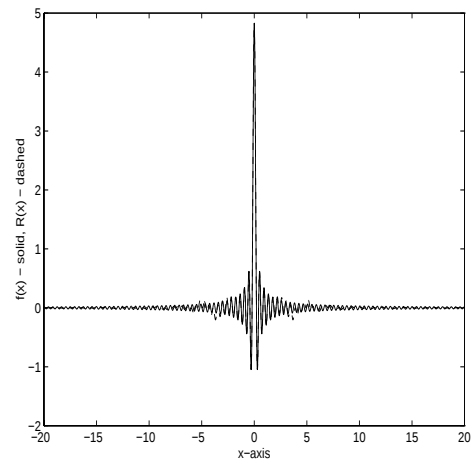


Figure 3: The function f and its reconstruction R .