ANALYSIS OF THE ADAPTIVE MATCHED FILTER ALGORITHM FOR CASES WITH MISMATCHED CLUTTER STATISTICS

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ABSTRACT

In practical radar applications of the adaptive matched filter algorithm, the covariance matrix for the clutter-plus-noise is typically estimated using data taken from range cells surrounding the cell under test. In a nonhomogeneous environment, this can lead to a mismatch between the mean of the estimated covariance matrix and the true covariance matrix for the range cell under test. Closed form expressions are provided, which give the performance for such cases. These equations are exact in some cases and provide useful approximate results in others. Performance depends on a small number of important parameters. These parameters describe which types of mismatches are important and which are not. Numerical examples illustrate how performance varies with each of the important parameters. Monte Carlo simulations are included which closely match the predictions of our equations.

1. INTRODUCTION

In most adaptive radar detection algorithms, the clutter-plus-noise in the cell under test is characterized using samples taken from range cells that neighbor the cell under test. This can lead to a mismatch between the true clutter-plus-noise statistics (in the cell under test) and those used to design the adaptive processing scheme. Such mismatches can occur in nonhomogeneous noise-plus-clutter cases. The purpose of this research is to develop analytical formulas that characterize the loss in performance due to this mismatch. Formulas of this type do not currently exist.

To understand why there may be a mismatch between the statistics of neighboring range cells, consider the case of ground clutter in airborne radar. The ground clutter returns corresponding to neighboring range cells are produced by reflections from different portions of ground. If very different types of objects reside on these different portions of ground, it is reasonable to expect that the clutter returns from these different range cells will be different. This behavior has been observed in measured data.

Consider a popular constant false alarm rate (CFAR) algorithm called the adaptive matched filter algorithm [1,2] with the test statistic

$$\Lambda_{AMF} = \frac{|s^H R_e^{-1} x|^2}{|s^H R_e^{-1} s|} \qquad (1)$$

In (1), the observed N dimensional complex vector x consists of either zero-mean complex Gaussian noise-plus-clutter [3] with covariance matrix R_t or signal plus zero-mean complex Gaussian noise-plus-clutter with covariance matrix R_t. The signal that is added to the noise-plus-clutter is κs where s is a unit length signal vector which is completely known and κ is an unknown complex constant. The magnitude of κ sets the signal-to-noise ratio. The denominator of (1) provides the correct normalization for CFAR (for cases without mismatch). While it would be desirable to use the true covariance R_t in (1) in place of R_e, R_t is not available. Instead an estimated covariance matrix R_e is used. R_e is obtained from the maximum likelihood (ML) estimate for the case where a set of independent reference vectors $x(k)$, $k = 1,...,L$ are available, all with the same distribution as the cell under text data x (even if this is not truly the case). Specifically, R_e is taken as

$$R_e = \sum_{k=1}^{L} x(k)\, x(k)^H \qquad (2)$$

which differs from the ML estimate by a scale factor. Note that in practice one frequently finds R_e is not close to a scaled version of R_t. In fact, the expected value of R_e may differ from LR_t which implies that even as $L \to \infty$, $(1/L)$ R_e will not converge to R_t. For the purpose of this paper we assume a mismatch in the reference data such that $(1/L)\, \mathrm{E}\{R_e\} = R_{sd} \neq R_t$. For simplicity we assume that the reference data vectors are independent and identically distributed (iid) with a zero-mean complex Gaussian distribution with covariance matrix R_{sd}. Also, the reference data are independent from the data from the cell under test.

In section II, we develop the distribution of the test statistic in (1) under some conditions. The conditions assume certain quantities are uncorrelated. If these quantities are correlated, our results are only approximate, but numerical investigations indicate that the approximations are very accurate. These conditions also assume that the value of a specific quantity, a generalization of what was called the loss factor in previous research [4], is known, but the probability density function (pdf) of the loss factor is found in Section III. Closed form expressions for the probability of false alarm and detection are provided in Section IV. Numerical evaluations of these expressions are provided in Section V to illustrate the effects of covariance matrix mismatch on performance. Conclusions are provided in Section VI.

2. DISTRIBUTION OF THE TEST STATISTICS

First apply a coordinate transform, which consists of multiplication by $R_{sd}^{-1/2}$, to the observed vector from the cell under test, the reference data vectors and to the signal vector. This transform whitens the reference data. Next, normalize the transformed signal vector so that it is again a unit vector. Call this transformed unit signal vector v. The transformed signal is taken to be $\beta v = [(s^H R_{sd}^{-1} s)^{1/2} \kappa] v$, so that signal-to-noise ratio is maintained. The important parameter $|\beta|^2$ is called the signal-to-secondary noise ratio (SSNR). It plays a role similar to the role the signal-to-noise ratio plays in cases without mismatch. Next, another transformation of coordinates, which consists of multiplication by $(v, B_v^H)^H$, is made to the cell under test vector, the reference data vectors and to the signal vector. Here B_v is a matrix whose *N-1* rows consist of a set of *N* dimensional vectors which span the space orthogonal to v. The overall matrix $(v, B_v^H)^H$ is taken to be unitary so that the signal and noise powers are preserved. Call the transformed vector for the cell under test $y = (d, z^T)^T$. Here *d* is a scalar that describes the component of the observed vector *x* which lies in the direction of *s*. On the other hand, *z* is an *N-1* dimensional vector describing the component of *x* that is orthogonal to *s*. The transformed reference vectors employ the notation $y(k) = (d(k), z(k)^T)^T$. Under the assumptions outlined, we find that (1) becomes

$$\Lambda_{AMF} = \frac{|d - \hat{r}_{zd}^H \hat{R}_z^{-1} z|^2}{|\hat{\sigma}_d^2 - \hat{r}_{zd}^H \hat{R}_z^{-1} \hat{r}_{zd}|} \tag{3}$$

where $\hat{r}_{zd} = \sum_{k=1}^{L} z(k) d(k)^H$, $\hat{\sigma}_d^2 = \sum_{k=1}^{L} d(k) d(k)^H$ and $\hat{R}_z = \sum_{k=1}^{L} z(k) z(k)^H$.

Now assume that *z* and *d* are uncorrelated. Then using the results from [5, pp. 113-118] and [6, pp. 27-29], it follows that the denominator of (3), when conditioned on *z, z(1),...z(L),* is a constant factor of *1/2* times a central chi-squared distributed random variable, with *2(L-N+1)* degrees of freedom. Such a random variable is denoted by $\chi^2_{2(L-N+1)}(0)$. Next, note that the term in the numerator of (3), inside the $|\;|^2$, is

$$y_r = d - \sum_{k=1}^{L} d(k) z(k)^H \hat{R}_z^{-1} z \tag{4}$$

Under either signal absent or signal present, the variance of y_r when conditioned on *z, z(1), … z(L)* is

$$Var\{y_r\} = \frac{1}{\rho} = Var\{d\} + z^H \hat{R}_z^{-1} z \tag{5}$$

where $Var\{d\} = v^H R_{sd}^{-1/2} R_t R_{sd}^{-1/2} v$. Next, we properly normalize (multiply by ρ) the numerator of (3) so it is the square of a unit-variance complex Gaussian. Conditioned on *z, z(1),...z(L),* another Theorem from [5, pp. 113-118] shows that $|\rho^{1/2} y_r|^2$ is a constant factor of *1/2* times a non-central chi-squared distributed random variable with *2* degrees of freedom, $\chi_2^2(\rho|\beta|^2)$, when signal is present. When signal is absent, the same holds true with $\beta = 0$. Thus, in summary, conditioned on *z, z(1),...z(L)* the test statistic in (3) is the ratio of a non-central chi-squared random variable with *2* degrees of freedom to ρ times a central chi-squared random variable with *2(L-N+1)* degrees of freedom as in

$$\Lambda_{AMF} \sim \frac{\chi_2^2(\rho|\beta|^2)}{\rho \chi^2_{2(L-N+1)}(0)} \tag{6}$$

The result in (6) requires that *z* and *d* to be uncorrelated to be exactly true. We demonstrate in Section 5 that results using (6) provide excellent approximation even when *z* and *d* are correlated.

3. DISTRIBUTION OF ρ

Consider the random variable

$$P = \frac{1}{z^H \hat{R}_z^{-1} z} = \frac{2\,\dfrac{z^H z}{z^H \hat{R}_z^{-1} z}}{2 z^H z} = \frac{s_1}{s_2} \tag{7}$$

Now, from (7), *P* is the ratio of two random variables. From [6] the first, s_1, is a central chi-squared random variable with $2(L-N+2)$ degrees of freedom and s_2 is independent of s_1. Note that *z* is complex Gaussian with zero mean and covariance matrix

$$R_{MM} = B_v R_{sd}^{-1/2} R_t R_{sd}^{-1/2} B_v^H \tag{8}$$

We define the eigenvalues of the R_{MM} to be $\Phi_1, \Phi_2, ... \Phi_{N-1}$.

It can be shown that s_2 has characteristic function $\prod_{j=1}^{N-1} (1 - 2it\Phi_j)^{-1}$, where $i = \sqrt{-1}$ and *t* is the frequency variable. By expanding this characteristic function into a partial fraction expansion and using a Fourier transform, one finds the pdf of s_2. For example, if no eigenvalues are repeated, then

$$\prod_{j=1}^{N-1} \frac{1}{1 - 2it\phi_j} = \sum_{j=1}^{N-1} \frac{b_j}{1 - 2it\phi_j} \tag{9}$$

with $b_j = \phi_j^{N-2} \prod_{k=1,k\neq j}^{N-1} (\phi_j - \phi_k)^{-1}$. Using well known Fourier transform results, the pdf for s_2 is

$$f_{S_2}(s_2) = \sum_{j=1}^{N-1} \frac{b_j}{2\phi_j} \exp(-\frac{s_2}{2\phi_j}) \qquad (10)$$

as long as s_2 is positive (the pdf is zero otherwise). If a particular $\phi_j = \phi$ is repeated r times, this leads to replacing r terms on the right hand side of (9) with (the new partial fraction expansion) $\sum_{j=1}^{r} \frac{B_j}{(1-2it\phi)^j}$ and thus these terms contribute

$$\sum_{j=1}^{r} B_j \frac{(s_2/\phi)^{j-1}}{2^j \Gamma(j)\phi} \exp(-\frac{s_2}{2\phi}) \qquad (11)$$

to (10) as long as s_2 is positive (the pdf is zero otherwise). Using (10) and (11) as appropriate one can find the pdf of s_2.

Now returning to (7), standard techniques for mapping of random variables give (f_X is the pdf of *X*)

$$f_P(p) = \int_{s_2=0}^{\infty} s_2 f_{S_1}(ps_2) f_{S_2}(s_2) ds_2 \qquad (12)$$

for $p \geq 0$. For example, if no eigevalues of R_{MM} are repeated (12) becomes

$$f_p(p) = \sum_{j=1}^{N-1} \frac{(L-N+2) b_j p^{L-N+1}}{\Phi_j (p + 1/\Phi_j)^{L-N+3}} \qquad (13)$$

If an eigenvalue ϕ is repeated r times then

$$\sum_{j=1}^{r} \frac{B_j (L-N+j+1)!\, p^{L-N+1}}{(j-1)!(L-N+1)!\phi^j (p+1/\phi)^{L-N+j+2}} \qquad (14)$$

should be used to replace the terms in (13) corresponding to the repeated eigenvalues. Using (13) and (14) allows one to find a closed form expression for the pdf of *P* in any given case. Then using mappings of random variables, it is straightforward to show that

$$f_\rho(\rho \mid H_0) = f_\rho(\rho \mid H_1) = \left(\frac{1}{1 - Var\{d\}\rho} \right)^2 f_P\left(\frac{\rho}{1 - Var\{d\}\rho} \right) \qquad (15)$$

for $0 < \rho < 1/Var\{d\}$, otherwise the pdf is zero.

4. PROBABILITY OF DETECTION AND FALSE ALARM

The distributions of ρ found in the last section lead to an easy way to derive the detection and false alarm probabilities. The approach is similar to that taken in [2] for cases without mismatch. Conditioned on ρ, we have already shown in (6) that our test statistic has exactly the same form as the one in [2, equation (27)], so by the same arguments given there we find that the detection probability conditioned on ρ is obtained from [2,equation (33)] as

$$P_{D|\rho} = 1 - \frac{1}{(1+\tau\rho)^{L-N+1}} \sum_{m=1}^{L-N+1} \frac{(L-N+1)!}{(L-N+1-m)!m!} (\tau\rho)^m G_m\left(\frac{|\beta|^2 \rho}{1+\tau\rho} \right) \qquad (16)$$

where $G_m(y) = \exp(-y) \sum_{k=0}^{m-1} \frac{y^k}{k!}$ as defined in [2]. In (16) τ is the threshold (3) is compared to. Using (15) and (16) gives the unconditional probability of detection as

$$P_D = \int_0^{1/Var\{d\}} P_{D|\rho} f_\rho(\rho \mid H_1) d\rho \qquad (17)$$

The false alarm probability follows from (16) and (17) with $\beta = 0$.

5. EFFECTS OF MISMATCH: NUMERICAL RESULTS

For the case with mismatch, (13) (15) and (17) imply that performance is sensitive to how much *Var{d}* and the eigenvalues of R_{MM} differ from unity[1].

Extensive numerical studies using (15) and (17) indicate that the probability of detection generally increases as *Var{d}* decreases, with the eigenvalues of R_{MM} being fixed. This is reasonable since *1/Var{d}* measures the signal-to-noise ratio of *d* and any signal energy will end up in *d*. A particular example of this is illustrated in Fig. 1 for cases with *N=2, L=4*, where calculations using (15) and (17) are shown to match Monte Carlo simulations results. When *Var{d}* is fixed, we see that increasing any eigenvalue, Φ_j causes a decrease in probability of detection. This is reasonable since each eigenvalue can be thought of as the noise-plus-clutter power in a particular one-dimensional subspace after imperfect whitening. Clearly increasing noise-plus-clutter power should lead to a decrease in performance. An example of this is shown in Fig. 2 for some *N=2, L=4* cases. Our studies also indicate that increasing *Var{d}* or any of the eigenvalues of R_{MM} leads to an increase in the probability of false alarm. This can be expected based on the interpretations of Var{d} and ϕ_j as being noise powers.

When *z* and *d* are uncorrelated, our results are exactly correct. When *z* and *d* are correlated, our equations are only approximately true, but numerical results show that they give very accurate approximations. Some typical examples are shown in Fig. 3 for *N=2, L=4* cases, for several values of

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[1] β, *N* and *L* affect performance in a similar way as for cases without mismatch, if β is viewed as SNR.

$r_{zd} = E\{zd^*\}$. We can see that the difference in probability of detection is very small in all cases. Through our study, we found that the maximum error is typically observed for the maximum value of r_{zd} (for the covariance matrix of $(d, z^T)^T$ to be positive definite, r_{zd} must be less than this maximum value). In a few cases, those with r_{zd} near its maximum value, the curves are a bit farther apart. A case with the maximum error we found in our study is shown in Fig. 4.

6. CONCLUSIONS

An analysis of the performance of the adaptive matched filter algorithm has been provided for cases where the data used to estimate the covariance matrix is not matched to the true covariance matrix of the data to be tested. Closed form expressions are given to estimate the probability of false alarm and detection. The equations indicate that performance depends on a few critical parameters. These parameters happen to be the eigenvalues of the covariance matrix of the observations after the imperfect whitening that occurs due to the covariance matrix mismatch, the signal-to-secondary-noise ratio, the observed vector size and the number of reference samples used to form the covariance matrix estimate.

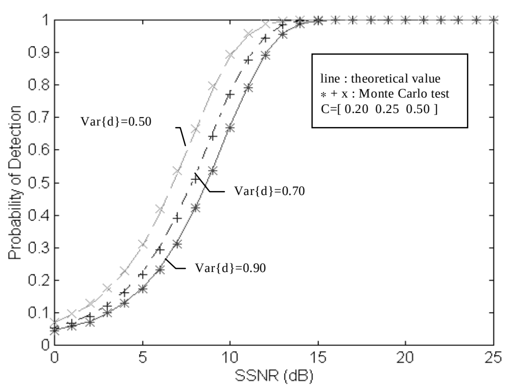

Fig. 1. Probability of detection variation with *Var{d}*

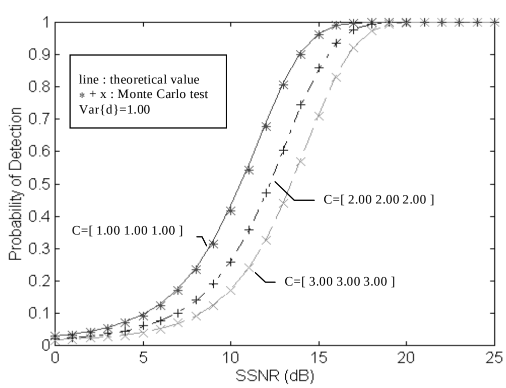

Fig.2. Probability of detection variation with $c = [\phi_1, \phi_2, \phi_3]$

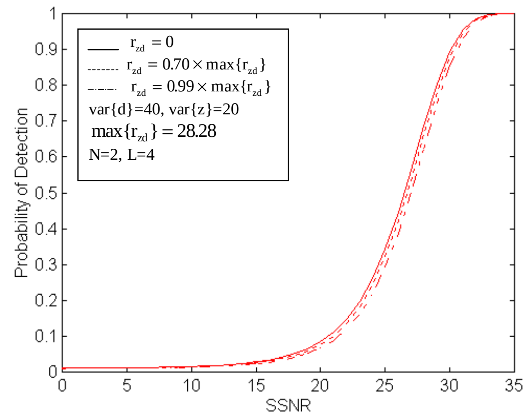

Fig. 3. Typical case when *d* and *z* are correlated.

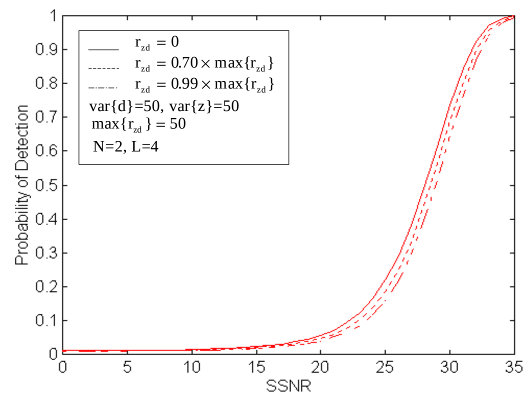

Fig. 4. Maximum error case when *d* and *z* are correlated.

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