ON FREQUENCY ESTIMATION FROM OVERSAMPLED QUANTIZED OBSERVATIONS

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ABSTRACT

The effect of sampling and quantization on frequency estimation for a single sinusoid is investigated. Asymptotic Cramér-Rao bounds (CRB) for 1-bit quantization and for non-ideal filters are derived, which are simpler to calculate than the exact CRB while still relatively accurate. It is further investigated how many bits should be used in quantization to avoid the problems of 1-bit quantization, and it turns out that 3-4 bits are enough. Finally, oversampled 1-bit quantization is investigated. It is determined how much the signal should be oversampled, and in addition $\Sigma\Delta$ modulators are investigated.

1. INTRODUCTION

This paper considers the classical problem of estimating the parameters of a single complex-valued sinusoid (cisoid) in additive Gaussian noise. The estimation is usually done digitally, and the paper therefore investigates the effects of digital processing, namely the effects of sampling, including anti-aliasing filters, and quantization.

The initial input signal to the system is a continuous time observed signal, which can be modeled as (for $t \in (-\infty, \infty)$)

$$x(t) = Ae^{i(\omega t+\phi)} + w(t) = s(t;\theta) + w(t) \tag{1}$$

where $w(t)$ is continuous time white Gaussian noise (WGN) with power σ^2. The noise power σ^2, the amplitude A ($A > 0$), the (angular) frequency ω, and the initial phase ϕ are all unknown. The parameter of main interest is ω. The information on ω in data is coupled to the information on the initial phase so ω and ϕ are gathered in the parameter vector $\theta = (\omega \;\; \phi)^T$ where T denotes transpose.

Prior to sampling, the noisy cisoid is transmitted through an analog anti-aliasing filter. We here assume that the signal has been stationary for so long time prior to the start of the sampling process, that we can disregard the transient response. Thus, if the anti-alising filter has frequency response $H(\omega)$ and the sampling time is T_s the sampled signal is (for $k = -\infty, \dots, 0, 1, \dots, \infty$)

$$x[k] = s[k;\theta] + v[k] = AH(\omega)e^{i(\omega T_s k+\phi)} + v[k]. \tag{2}$$

Here and in the sequel all quantities associated with the continuous time signal are denoted by $(\cdot)$ and all quantities associated with the discrete time signal by $[\cdot]$. In (2), $v[k]$ is additive Gaussian noise, which is not necessarily white.

After sampling, the signal is quantized, i.e., rounded to one of a finite number of levels. If the quantization is very fine, e.g., 12 bits precision, the quantization can be disregarded or treated as another source of additive noise. However, some applications deal with very high frequency signals and fine quantization is impossibly or economically infeasible. We therefore consider coarse quantization, in particular single-bit quantization, with observations given by,

$$x_+[k] = \mathrm{sign}(\Re(x[k])) + i\,\mathrm{sign}(\Im(x[k])) \tag{3}$$

where $\Re(\cdot)$ and $\Im(\cdot)$ denote the real and imaginary parts of the quantity between the parentheses, respectively, and $\mathrm{sign}(\cdot)$ denotes the sign function, i.e. $\mathrm{sign}(x) = 1$ for $x \geq 0$ and $\mathrm{sign}(x) = -1$ for $x < 0$. The advantage of 1-bit quantization is the simple implementation, which has made it popular in, for example, $\Sigma\Delta$-modulators. For the present application, 1-bit quantization also has the advantage that no gain control is needed and, as will be seen below, that very efficient algorithms for processing of 1-bit samples can be made.

Since we are mainly interested in the increase in variance due to quantization and sampling, we will often use *relative variance* or *relative CRB*, meaning the variance respectively the CRB relative to $\mathrm{CRB}(\hat{\omega}) = 6\sigma^2/(A^2N(N^2-1))$ corresponding to the CRB for un-quantized data and ideal anti-aliasing filters at sampling rate $T_s = 1$ [3].

2. EFFECTS OF QUANTIZATION

The following theorem was derived in [1].

Theorem 1 *[1] Consider the sampled signal (2) for an ideal anti-aliasing filter $H(\omega) = 1$. Then the Fisher information matrix for the parameters $\theta = (\omega \; \phi)^T$ corresponding to the 1-bit quantized observations $\{x_+[0], \dots, x_+[N-1]\}$ in (3) is given by*

$$\mathbf{I}(\theta) = 2\frac{2}{\pi}\left(\frac{A}{\sigma}\right)^2 \sum_{k=0}^{N-1} \begin{bmatrix} k^2 & k \\ k & 1 \end{bmatrix} \kappa(\omega k + \phi; A/\sigma) \tag{4}$$

where $0 < \kappa(\varphi; s) \leq 1$ is a function with period $\pi/2$, given by

$$\kappa(\varphi; s) = \frac{\exp\left(-2s^2\cos^2\varphi\right)}{1-\mathrm{erf}^2(s\cos\varphi)}\sin^2\varphi + \frac{\exp\left(-2s^2\sin^2\varphi\right)}{1-\mathrm{erf}^2(s\sin\varphi)}\cos^2\varphi. \tag{5}$$

In contrast to the case of no quantization, the Fisher matrix and CRB cannot in general be summed explicitly to give a simple formula. However, as $\kappa(\varphi; s)$ is periodic with period $\pi/2$, the function $\kappa(\omega k+\phi; A/\sigma)$ is independent of k for ω an (integer) multiple

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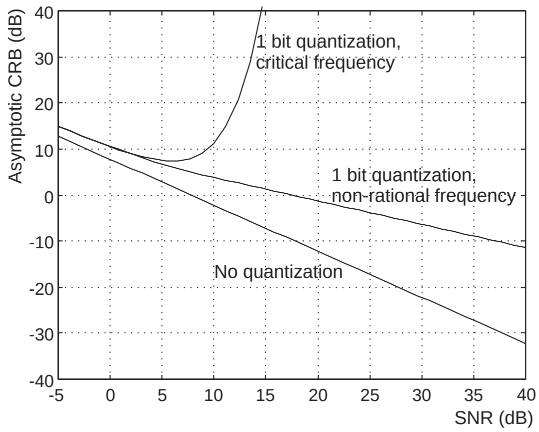

Figure 1: Asymptotic CRB as a function of SNR. A *critical frequency* is a frequency with ω an (integer) multiple of $\pi/2$.

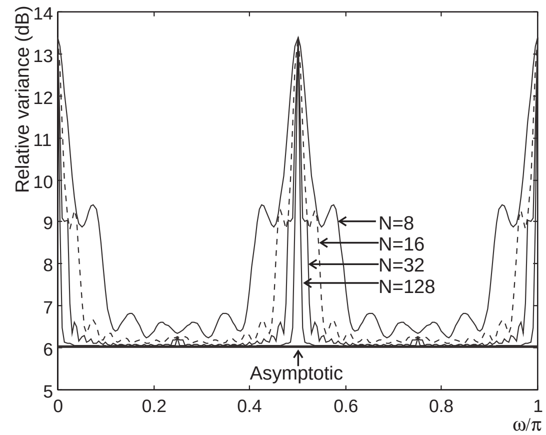

Figure 2: Relative variance versus frequency for varying N. The SNR is 10dB.

of $\pi/2$, so that we get

$$\mathrm{CRB}(\hat{\omega})|_{\omega=n\pi/2} = \frac{6\sigma^2}{A^2N(N^2-1)}\frac{\pi}{2\kappa(\phi;A/\sigma)}. \qquad (6)$$

where $\mathrm{CRB}(\hat{\omega})$ is the (1,1)-element of $\mathbf{I}(\theta)^{-1}$. This CRB is strongly dependent on the phase ϕ. Since a realistic assumption is that the phase is a uniform random variable, a more interesting CRB is obtained by averaging (6) over the phase [2]. For other values of ω the following theorem can be used

Theorem 2 *[2] Suppose that ω/π is irrational. Then the asymptotic CRB at ω for 1-bit sampling is given by*

$$\mathrm{AsCRB}(\hat{\omega}) = \lim_{N\to\infty} N^3\mathrm{CRB}(\hat{\omega}) = \frac{6\sigma^2}{A^2}\frac{\pi}{2\bar{\kappa}(A/\sigma)} \qquad (7)$$

where
$$\bar{\kappa}(A/\sigma) = \frac{1}{2\pi}\int_0^{2\pi}\kappa(\varphi;A/\sigma)d\varphi. \qquad (8)$$

Notice that the asymptotic CRB is not dependent upon the phase.

By numerical calculation it appears that $\bar{\kappa}(s)s \to K$ for $s \to \infty$ where $K \approx 1.28$. We have not tried to prove this formally, but have observed that for SNR > 10 dB the relation is accurate within a few percent. This gives the following approximation to the CRB (for ω/π irrational)

$$\mathrm{CRB}(\hat{\omega}) \approx \begin{cases} \frac{7.36}{\sqrt{\mathrm{SNR}}\,N^3} & \text{for high SNR} \\ \frac{9.42}{\mathrm{SNR}\,N^3} & \text{for low SNR} \end{cases}. \qquad (9)$$

Figure 1 shows the asymptotic CRB (7) versus SNR. For low SNR the only effect of quantization is an increase of the CRB by a factor of $\pi/2$ [2]. However, for large SNR, and at frequencies with ω a multiple of $\pi/2$, which we will call *critical frequencies,* the CRB *increases* with SNR. This might seem surprising, but is not unusual for coarse quantization. For the non-critical frequencies, the CRB decreases with increasing SNR, but not as fast as for no quantization.

The asymptotic CRB can be used as a good approximation for non-critical frequencies and $N \geq 16$, as seen from Figure 2. At the critical frequencies, on the other hand, (6) averaged over the phase can be used. From this equation it can be seen that the 'height' of the peaks at the critical frequencies does not decrease with increasing N, whereas from the figure it is revealed that the *width* decreases (the width is approximately inversely proportional to N). This means that for large N the effect of the critical frequencies will be minimal, since even a slight jitter in frequency will make the probability of hitting a frequency inside the peak very small. But for small N the critical frequencies is a fact that has to be taken seriously into consideration.

For quantization with a higher number of bits, the problem to be solved is the optimization of the quantization levels. This could be solved by calculating the CRB and minimizing the CRB with respect to the quantization levels. However, as the CRB for 1-bit quantization is already quite complicated, we have not attempted to calculate the CRB for higher number of bits. Instead we have pursued a more heuristic way. Following [5], we selected the quantization levels by minimizing the mean square error between the continuous signal and the quantized signal. Here we limited attention to a symmetric, uniform quantizer with B bits. Furthermore, the real and imaginary part of the signal are quantized independently. Specifically, if we let $Q(x;\Delta)$ be the output of the quantizer, we can then find Δ by minimizing the mean square difference between the (real part of the) signal and its quantized value, averaged over all phase values,

$$\Delta_{\mathrm{opt}} = \arg\min_{\Delta}\int_0^{2\pi} E\left[|Q(A\sin(x)+v;\Delta)-(A\sin(x)+v)|^2\right]dx \qquad (10)$$

where v is a zero-mean Gaussian random variable of variance $\sigma^2/2$. The optimization (10) can be done by calculating the integral and expectation numerically, and performing a 1-dimensional numerical peak location. In general the optimal value of Δ depends on the SNR. For high SNR, and a large number of bits, an ad-hoc way of chosing Δ is to utilize the full scale of the quantizer, i.e. to put $\max_\varphi A\cos\varphi = \max_x Q(x;\Delta)$, giving $\Delta \approx A/(2^{B-1}-1/2)$.

3. EFFECTS OF SAMPLING ON CRB

As was shown in the previous section, when using coarse quantization it is necessary to oversample the signal in order to avoid the variance increase at certain frequencies. It is therefore also necessary to consider the actual anti-aliasing filters used prior to sampling.

We consider a fixed rational low-pass filter $H(s)$, i.e., such that $H(s)$ is a rational function in s with the degree of the numerator larger than that of the denominator. This means that the limit $\lim_{\omega\to\infty}|H(\omega)|^2 = 0$. We assume that $H(s)$ has (circular) cut-off frequency 1, and we vary the cut-off frequency ω_c by considering $H(s/\omega_c)$.

The continuous time WGN process $w(t)$ in (1) is transmitted through the filter $H(s/\omega_c)$, with output $v(t)$. The process $v(t)$ then is a Gaussian random process with spectral density $S(s/\omega_c) = H(s/\omega_c)H(-s^*/\omega_c)^*$ (where $*$ denotes conjugate) and correlation function $R(t)$. The process $v(t)$ is sampled equidistantly with sampling interval $T_s = 2\pi/\omega_s$, giving the discrete time noise $v[k]$. It is clear that the correlation function of $v[k]$ is the sampled correlation function of $v(t)$, that is $R[k] = R(kT_s)$. We can therefore find the spectral density $S[z]$ of $v[k]$ by residue calculus. The following asymptotic (as $N \to \infty$) result holds true,

Theorem 3 *[2] Consider the problem of estimating ω in (2) with $T_s = 2\pi/\omega_s$ and $H(s/\omega_c)$ being a continuous time anti-aliasing filter with cut-off frequency ω_c and $\lim_{\omega\to\infty}|H(\omega)|^2 = 0$. Then, the asymptotic CRB is*

$$\mathrm{AsCRB}(\hat{\omega}) = \lim_{N\to\infty} N^3\mathrm{CRB}(\hat{\omega}) = \frac{6\sigma^2 S[\omega T_s]}{A^2|H(\omega/\omega_c)|^2 T_s^2}. \quad (11)$$

The quotient $|H(\omega)|^2A^2/S[\omega T_s]\sigma^2$ can be interpreated as the local SNR at ω. The asymptotic CRB gives some insight into the effect of sampling and anti-aliasing filters, and it is straightforward to calculate. In contrast to asymptotic CRB for 1-bit sampling, it has, however, a limited accuracy if N is small, the cut-off frequency ω_c is small, or the sampling frequency ω_s is large.

For the asymptotic CRB, the optimization of sampling can partly be done analytically. First, we have

Theorem 4 *[2] For fixed ω, $\mathrm{AsCRB}(\hat{\omega})$ is an increasing function of ω_c for the class of Butterworth filters.*

This can be interpreted so that for large N, the cutoff frequency ω_c should be chosen as small as possible. The CRB can then be found by

Theorem 5 *[2] Let $H(s)$ be a general, rational filter, and let $m > 0$ be the difference between the degree of the numerator and the denominator of $H(s)$. Then*

$$\lim_{\omega_c\to 0}\mathrm{AsCRB}(\hat{\omega}) = -\frac{6\sigma^2 T_s^{2m-3}\omega^{2m}}{A^2 2^{2m}(2m-1)!}\cot^{(2m-1)}\left(-\frac{\omega T_s}{2}\right). \quad (12)$$

This expression can be used as a good approximation for small ω_c. As far as we know, no closed form expression exists for the n'th derivative of $\cot(\cdot)$, but for small n compact formulas can easily be found. Once the derivative of $\cot(\cdot)$ has been found, T_s can be easily optimized numerically.

The problem for 1-bit quantization is that the variance increases dramatically at the critical frequencies. One way to avoid these critical frequencies is to oversample with at least a factor 4, so that only the spectrum between two critical frequencies is used. Since one of the critical frequencies is 0, the signal also has to be frequency shifted prior to sampling, so that 0 frequency is positioned in the middle between two critical frequencies. This can be done by applying a frequency shift of $1/8$ of the sampling frequency prior to sampling.

¿From Figure 2 it can be seen that the peaks around the critical frequencies do not have zero width. We therefore chose the oversampling factor so that the maximum and minimum frequencies will be at the *second local minimum* from the peak. For $N = 16$ this minimum is situated at approximately $\omega/\pi = 0.06$. This means that the usable part of the spectrum is situated in the range $\omega/\pi \in [0.06, 0.44]$. To map the whole range $\omega/\pi \in [-1, 1]$ into this range, the signal has to be oversampled by a factor 5.25 and shifted in frequency by 1.3125π, i.e., $1/8$ of the sampling frequency. For $N = 32$ the an oversampling of 4.5 times is needed, and for $N = 8$ an oversampling of 8 times is needed.

While the oversampling factor depends on the number of samples used, the frequency shift is always $1/8$ of the sampling frequency. This shift frequency could be obtained by using analog frequency doublers or frequency dividers.

The specific anti-aliasing filter and its cutoff frequency is not very critical due to the large oversampling factor. We find that a Butterworth filter of order at least two will do. The CRB has a minimum for a cutoff frequency of approximately π, but is not very sensitive to the cutoff frequency. The variation is a few percentage in a wide range around π.

An alternative to using oversampling by itself is to use an oversampled $\Sigma\Delta$ modulator. The advantage of using a $\Sigma\Delta$ modulator is that the frequency shift is not needed. We found that when a $\Sigma\Delta$ modulator is used, the increase in variance at 0 frequency disappears. However, a $\Sigma\Delta$ modulator increases variance at high frequencies, and an oversampling factor of at least 4 should therefore be used. The advantage of using the $\Sigma\Delta$ modulator compared to plain oversampling is then that the frequency shift is not needed. However, the gain control for the $\Sigma\Delta$ modulator needs to be rather accurate. We found that the amplitude of the sinusoid into the $\Sigma\Delta$ modulator should be around 1.

The same processing as for plain oversampling can be used to the output of the $\Sigma\Delta$ modulator.

4. FREQUENCY ESTIMATION

For white noise and un-quantized data the maximum likelihood estimator (MLE) is well known given by the location at which the periodogram $P(\omega)$ attains its maximum [3]. Under an assumption on high SNR, a formula for the argument of $\max P(\omega)$ is [2]

$$\hat{\omega} = \frac{12}{N^2(N^2-1)}\sum_{k=1}^{N-1} k(N-k)\angle[\hat{R}[k]] \quad (13)$$

where $\hat{R}[k]$ (for $k = 0, \ldots, N-1$) is the biased autocovariance estimator, and $\hat{R}[-k] = \hat{R}[k]^*$. A simplification of (13) is to truncate the sum after $M < N-1$ terms and replace the parabolic window with an arbitrary window $V[k]$, see [2]. The integer M roughly determines the trade-off between numerical complexity and statistical accuracy. For $M > 1$ a direct implementation of an estimator based on $\angle[\hat{R}[k]]$ has to be combined with some phase unwrapping procedure. Alternatively, they can be rewritten in differential form.

For 1-bit quantized data it no longer holds true that the estimated autocorrelation $\hat{R}[m]$ is close to $R[m] = Ex[k]x^*[k-m]$, where $R[m] = A^2e^{i\omega m} + \sigma^2\delta_{m,0}$. In fact, for large N and small SNR it holds that $\hat{R}[0] \simeq 2$ and $\hat{R}[m] \simeq 4A^2e^{i\omega m}/\sigma^2\pi$, for $m = 1, \ldots, M$ [4]. Thus, a correlation based estimator provide approximately unbiased estimates for SNR slightly above its SNR-threshold. For large SNR the estimate no longer will be unbiased. An expression for the asymptotic bias (as SNR $\to \infty$, for a general $V[k]$) was derived in [1].

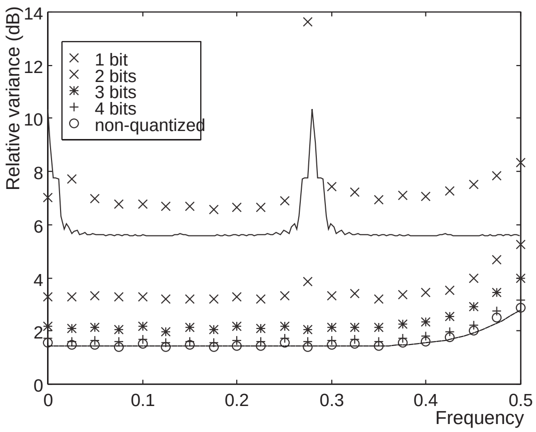

Figure 3: Variance versus frequency for $N = 32$ and SNR=12 dB for 4-th order *Butterworth* filters (see, table 1 in [2]). The number of ensembles for each frequency was 10000 with random phase.

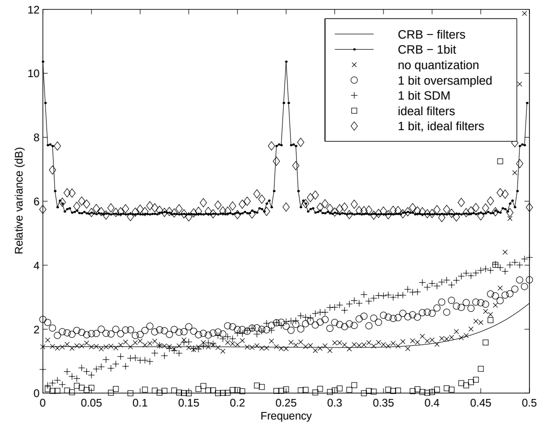

Figure 4: Variance versus frequency for $N = 32$ and SNR=12 dB for 4-th order Butterworth filters (see, table 1 in [2]). The number of ensembles for each frequency was 10000 with random phase.

5. SIMULATION RESULTS

Effects of Number of Bits on Variance Figure 3 show the variance of the frequency estimate for $N = 32$ samples for different number of bits used in the quantization. As estimator we used in all cases an FFT based estimator with 4 times zero-padding, and peak-finding by triple parabolic interpolation. As anti-aliasing filters 4-th order Butterworth filters were used with cut-off frequencies and sampling frequencies selected from table 1 in [2].

We can make several observations from the figures. First, we note that the dramatic increase in variance predicted by the CRB around certain frequencies for 1-bit quantization also show up in simulations (see '1 bit, ideal filters' in Figure 4). However, right on the critical frequencies (e.g., 0.25), the variance is lower than the CRB, while it is larger right next to the critical frequency. This is due to the fact that there is a strong bias towards the critical frequencies, so exactly these frequencies have a low variance. In Figure 3 the peak is not at exactly 0.25, since a slight oversampling is used for non-ideal filters.

Already by using 2 bits, the phenomenon almost disappears, and by using 4 bits the results are indistinguishable from the unquantized results. The quantization levels where optimized according to (10), but using $\Delta \approx A/(2^{B-1} - 1/2)$ gives almost the same results. Thus, 4 bits seems to be a good choice for quantization. Even if the gain control is wrong by a factor 2, the performance is no worse than for 3 bits, which only decreases performance by a fraction of a dB.

1-bit Quantization and Oversampling Here, we consider the correlation based estimators (13) called approximate MLE (AMLE).

Figure 4 shows the performance of the AMLE with or without 1-bit quantization and oversampling. It is seen that the AMLE reaches the CRB both for ideal filters and for Butterworth filters, except at frequencies near 0.5, where the phase unwrapping causes problems. The problem is less for the differential implementation, and can also be avoided by increasing the sampling frequency. However, the latter also increases either complexity or variance, depending on whether $T = N \cdot T_s$ or N is fixed.

For 1-bit quantization and oversampling, for $N = 32$ an oversampling factor of 4.5 is used together with $1/8$ frequency shift. For the oversampled $\Sigma\Delta$ modulator (SDM), also 4.5 times oversampling is used, but without a frequency shift. The number of samples used is $4.5 \cdot 32 = 144$. Using 128 samples are advantageous for hardware implementation, but gives a slightly higher variance [2]. It can be seen that the $\Sigma\Delta$ modulator gives a lower variance than plain oversampled 1-bit sampling at low frequencies, but a higher variance at high frequencies.

6. CONCLUSIONS

The asymptotic CRB on the variance of any consistently estimated frequency based on 1-bit quantized observations is derived. It is shown that for *critical frequencies* the CRB increases with increasing SNR. On a short distance from the critical frequencies the CRB decreases with increasing SNR.

It is also shown that the effects of quantization are practically negligible employing a quantizer with 3 or 4 bits.

An alternative to increasing the number of bits of the quantizer, is to use 1-bit quantization employing a sampling rate beyond Nyquist. A closed form expression for the asymptotic CRB on the variance of estimated frequency is derived based on observations prefiltered by a non-ideal anti-aliasing filter. It is shown that for reasonable sample sizes, a combination of oversampling a factor 4 combined with frequency shifting by $1/8$ of the sampling frequency fully eliminates the effects of quantization, even when employing a low-order anti-aliasing filter.

7. REFERENCES

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