OPTIMAL PHASE PARAMETER ESTIMATION OF RANDOM AMPLITUDE LINEAR FM SIGNALS USING CYCLIC MOMENTS

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ABSTRACT

This paper considers the problem of estimating the phase parameters of a linear FM signal which is modulated by a random process and is embedded in additive noise. In particular, we consider the use of cyclic moments and derive variance expressions for the phase parameter estimates for all values of the lag parameter of the second order cyclic moment, τ. It is seen that the accuracy of the phase parameter estimates depends greatly on τ. This allows the definition of an optimal value of τ, in the sense that it minimises the phase parameter estimation variance.

1. INTRODUCTION

Signals which have been subject to an amplitude modulation and/or a frequency modulation are frequently encountered in applications such as radar and sonar. In the presence of additive noise, the discrete-time model of such signals is

$$X_t = A_t s_t + W_t = A_t e^{j\phi_t} + W_t,\ t = 0, 1, \ldots, T-1. \quad (1)$$

We will be considering a special case of this signal model where,

A1. the signal phase can be expressed as a quadratic function of time,

$$\phi_t = a_1 t + a_2 t^2, \quad t = 0, 1, \ldots, T-1, \qquad (2)$$

A2. A_t is a real stationary i.i.d. random process with mean μ_a and finite variance σ_a^2.

A3. W_t is a complex stationary white Gaussian random process, independent of A_t, with variance σ_w^2.

Given observations, x_t, $t = 0, 1, \ldots, T-1$, from the signal model (1), we wish to estimate the phase parameters, a_1 and a_2.

Under the assumptions **A1-A3**, the signal model (1) is an appropriate model for radar and sonar return signals. In radar, target returns often exhibit non-linear phase characteristics due to the varying radial velocity of the target [9]. A constant radial velocity results in a linear phase shift, while a constant radial acceleration results in a quadratic phase shift. Thus by estimating the phase parameters of the return signal it is possible to obtain information about the velocity and acceleration of the target. The random amplitude is a result of the changing orientation of the target and is termed time-selective fading [12]. A similar situation exists in sonar where target motion and ocean effects give rise to signals of the type shown in (1) [11].

Most techniques for estimating the phase parameters of FM signals assume a constant amplitude i.e. $A_t \equiv A, t = 0, 1, \ldots, T-1$. Techniques which employ this assumption include the polynomial phase transform (PPT) [7], the integrated generalised ambiguity function [1], the polynomial Wigner-Ville distribution [2] and the least-squares method [5]. The random amplitude case has received substantially less attention. In [6] techniques were proposed for differentiating between constant and random amplitude polynomial phase signals. Boashash and Ristic showed that the Wigner-Ville trispectrum can be used to estimate the instantaneous frequency of random amplitude linear FM signals [3]. Shamsunder *et al.* proposed a phase parameter estimation procedure based on cyclic moments [10], but did not investigate the statistical properties of the estimates obtained. In this paper we present a statistical analysis of the estimates of the phase parameters, a_1 and a_2 obtained from the cyclic moments and establish the conditions under which these estimates are optimal. In particular, it is shown that the choice of lag parameter, τ, greatly affects the accuracy of the phase parameter estimates.

The paper is structured as follows. Section two presents a review of the estimation procedure proposed in [10]. In section three the variance expressions for the phase parameter estimates are presented. From these results the optimal value of τ is found. The computed variances are confirmed using numerical simulations. The paper concludes with a discussion of the results obtained.

2. CYCLIC MOMENTS

In order to illustrate the use of cyclic moments for estimating the phase parameters of the signal (1) we initially set $W_t \equiv 0$, $t = 0, 1, \ldots, T-1$. The second order moment of the signal X_t is defined as,

$$m_{2x}(t;\tau) \stackrel{\triangle}{=} E\, X_t^* X_{t+\tau}. \qquad (3)$$

Under the assumptions **A1** and **A2**, the second order moment is a constant amplitude sinusoid. The periodicity of the second order moment permits the generalised Fourier series expansion,

$$m_{2x}(t;\tau) \quad = \sum_{\alpha \in \mathcal{A}_{2x}} \mathcal{M}_{2x}(\alpha;\tau) e^{j\alpha t}, \qquad (4)$$

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$$\mathcal{M}_{2x}(\alpha;\tau) \quad = \quad \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} m_{2x}(t;\tau) e^{-j\alpha t}, \qquad (5)$$

where $\mathcal{A}_{2x} = \{\alpha : \mathcal{M}_{2x}(\alpha;\tau) \neq 0\}$. The generalised Fourier series coefficients, $\mathcal{M}_{2x}(\alpha;\tau)$, are called the second order cyclic moments and the signal (1) is said to exhibit second order cyclostationarity. The phase parameter, a_2, can then be found as

$$a_2 = \frac{\arg\max_{\alpha}\{|\mathcal{M}_{2x}(\alpha;\tau)|^2\}}{2\tau} \qquad (6)$$

The lower order phase parameter, a_1, can then be found by demodulating the original signal with the calculated value of a_2 and taking the cyclic mean of the demodulated signal,

$$\mathcal{M}_{1x^{(1)}}(\alpha) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} E\, X_t^{(1)} e^{-j\alpha t}, \qquad (7)$$

where $X_t^{(1)} = X_t e^{-ja_2t^2}$. The phase parameter is found as the peak of the cyclic mean, $\mathcal{M}_{1x^{(1)}}(\alpha)$. It is important to note that the phase parameters must satisfy the criteria

$$|a_m| \leq \frac{\pi}{m!\tau^{m-1}}, \quad m = 1, 2. \qquad (8)$$

We consider now observations, $x_t, t = 0, 1, \ldots, T-1$, obtained from the full signal model (1) under assumptions **A1**-**A3**. From this single realisation we wish to estimate the phase parameters, a_1 and a_2. This can be done using the following estimate of the second order cyclic moment,

$$\hat{\mathcal{M}}_{2x}(\alpha;\tau) = \frac{1}{T} \sum_{t=0}^{T-\tau-1} x_t^* x_{t+\tau} e^{-j\alpha t} \qquad (9)$$

The cyclic mean is estimated in an analogous manner. These cyclic moment estimates are asymptotically consistent [4]. The phase parameter estimation procedure is then

1. Estimate a_2 as

$$\hat{a}_2 = \frac{\arg\max_{\alpha\neq 0}\{|\hat{\mathcal{M}}_{2x}(\alpha;\tau)|^2\}}{2\tau}. \qquad (10)$$

2. Form the demodulated signal,

$$x_t^{(1)} = x_t e^{-j\hat{a}_2 t^2}, \quad t = 0, 1, \ldots, T-1. \qquad (11)$$

3. Estimate a_1 as

$$\hat{a}_1 = \arg\max_{\alpha\neq 0}\{|\hat{M}_{1x^{(1)}}(\alpha)|^2\}. \qquad (12)$$

In the following section the variance of the estimates will be established for all values of the lag parameter, τ.

3. STATISTICAL ANALYSIS

This section is devoted to an analysis of the variance expressions obtained for the phase parameter estimates, (10) and (12). This analysis is asymptotic in the sense that it is assumed that T is large and that σ_a^2 and σ_w^2 are small compared to μ_a. The proofs for the variance expressions can be found in the appendix.

Under assumptions **A1**-**A3** the variances of the phase parameter estimates are

$$\begin{aligned} &\mathrm{var}\{\hat{a}_2\} \approx \qquad (13)\\ &\frac{3}{2\tau^2 L(L^2-1)}\left[2 + \frac{\sigma_w^2 + 2\sigma_a^2}{\mu_a^2} \right.\\ &\left. - \frac{2(L-\tau)(L^2-2L\tau-2\tau^2)}{L(L^2-1)} u_{T-2\tau}\right] \frac{\sigma_w^2}{\mu_a^2} \end{aligned}$$

$$\begin{aligned} &\mathrm{var}\{\hat{a}_1\} \approx \qquad (14)\\ &\frac{3T^2}{2\tau^2 L(L^2-1)}\left[2 + \frac{\sigma_w^2 + 2\sigma_a^2}{\mu_a^2} + \frac{4\tau^2 L(L^2-1)}{T^5} \right.\\ &\left. - \frac{2(L-\tau)(L^2-2L\tau-2\tau^2)}{L(L^2-1)} u_{T-2\tau}\right] \frac{\sigma_w^2}{\mu_a^2} \end{aligned}$$

where $\tau = 1, \ldots, T-2$,[1] $L = T - \tau$ and $u_t = 1$, $t \geq 0$. Inspection of (14) and (15) reveals the following points,

- the variances of the estimates (10) and (12) display similar characteristics as the lag parameter, τ, varies. This is to be expected since the accuracy of $\hat{a}_2$ will directly affect the accuracy of $\hat{a}_1$.
- the variances increase as the variance of the modulating process and/or the variance of the additive noise increase.

A closer examination of the variance expressions will now be conducted. We plot the variance of each phase parameter estimate against τ for the case $T = 128$, $\mu_a = \frac{5}{4}$, $\sigma_a^2 = \frac{1}{8}$, $\sigma_w^2 = \frac{1}{3}$ and A_t is an i.i.d. Rayleigh random process. The plots are shown in Figures 1 and 2. The computed variances are compared with estimated variances obtained using 1000 realisations of the signal model (1) with $a_1 = \frac{1}{8}$ and $a_2 = \frac{1}{4096}$.

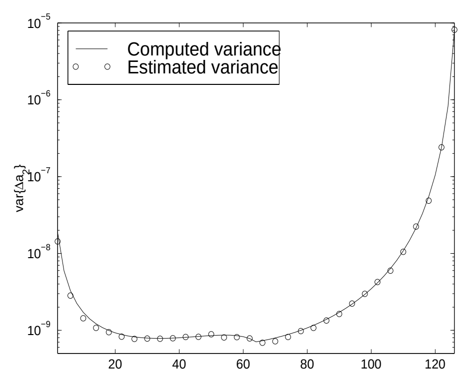

Figure 1: Variance of $\Delta\hat{a}_2$.

From the figures we see that both of the variances are minimised when $\tau = \frac{T}{2} = 64$. It can also be seen that the computed variances correspond closely to the estimated variances. It can be shown that for general values of T, the situation observed here holds and the optimal value of τ is $\frac{T}{2}$, provided T is even.

[1] Only positive values of τ are considered as only the absolute value of τ is of importance i.e. estimates obtained for $\tau = 1$ and $\tau = -1$ are statistically equivalent.

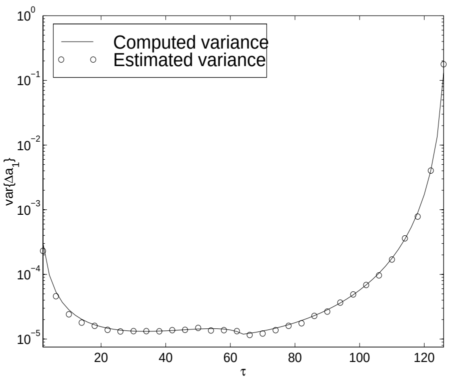

Figure 2: Variance of $\Delta\hat{a}_1$.

The variances of the phase parameter estimates $\hat{a}_2$ and $\hat{a}_1$ are then $O(T^{-5})$ and $O(T^{-3})$ respectively. In the event that T is odd, the optimal value of τ is $\frac{T+1}{2}$.

4. DISCUSSION

In the above sections the variances of the phase parameter estimates obtained from the cyclic moments were derived. It was shown that the value of the lag parameter, τ, chosen to compute the estimated second order cyclic moment, (9), greatly affects the accuracy of the phase parameter estimates, (10) and (12). In particular, it can be seen from (14) and (15) that the variances will be minimised when we choose $\tau = \frac{T}{2}$. This is the same result as that obtained when using the PPT to estimate the phase parameters of constant amplitude linear FM signals in additive white Gaussian noise. This correspondence is perhaps not that surprising, as the estimate of the second order cyclic moment is nearly identical to the second order PPT. The variance expressions were verified using numerical simulations, as shown in Figures 1 and 2.

Future work will extend the signal model considered to higher order polynomial phase signals and investigate the effect of correlation in the random modulating process.

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6. APPENDIX: DERIVATION OF VARIANCES

In this appendix the variance expressions (14) and (15) are derived. Throughout the analysis we will refer to the signal model and assume that **A1**-**A3** are satisfied.

6.1. Variance of $\hat{a}_2$

The second order cyclic moment of the sequence $\mu_a s_t$ is given by

$$\hat{\mathcal{M}}_{2\mu s}(\alpha) = \frac{\mu_a^2 e^{j\psi(\tau)}}{T} \sum_{t=0}^{L-1} e^{-j(\alpha - 2a_2\tau)t}, \qquad (15)$$

where $\psi(\tau) = a_1\tau + a_2\tau^2$. The perturbation due to the presence of the random amplitude modulation and additive noise is

$$\begin{aligned} \Delta\hat{\mathcal{M}}_{2\mu s}(\alpha) &= \hat{\mathcal{M}}_{2x}(\alpha) - \hat{\mathcal{M}}_{2\mu s}(\alpha) & (16)\\ &= \frac{1}{T}\sum_{t=0}^{L-1} s_t^* s_{t+\tau}\left[\left(A_t + \frac{W_t^*}{s_t^*}\right)\right.\\ &\quad \times \left.\left(A_{t+\tau} + \frac{W_{t+\tau}}{s_{t+\tau}}\right) - \mu_a^2\right] e^{-j\alpha t} & (17)\\ &= \frac{e^{j\psi(\tau)}}{T}\sum_{t=0}^{L-1} \epsilon_{t,\tau} e^{-j(\alpha - 2a_2\tau)t} & (18) \end{aligned}$$

The peak of $f_{2\mu s}(\alpha) = |\hat{\mathcal{M}}_{2\mu s}(\alpha)|^2$ occurs at $\alpha_0 = 2a_2\tau$. A first order perturbation analysis shows that the shift in the location of the peak can be approximated as [8]

$$\Delta\alpha_0 \approx -\frac{\dfrac{\partial \Delta f_{2\mu s}(\alpha_0)}{\partial \alpha}}{\dfrac{\partial^2 f_{2\mu s}(\alpha_0)}{\partial \alpha^2}} \qquad (19)$$

It can then be shown that

$$\frac{\partial^2 f_{2\mu s}(\alpha_0)}{\partial \alpha^2} = -\frac{\mu_a^4}{6T^2} L^2(L^2-1) \qquad (20)$$

and

$$\frac{\partial \Delta f_{2\mu s}(\alpha_0)}{\partial \alpha} \approx \frac{\mu_a^2 L}{T^2}\,\mathrm{Im}\left\{\sum_{t=0}^{L-1}(L-1-2t)\epsilon_{t,\tau}^*\right\} \qquad (21)$$

Therefore

$$\Delta\hat{a}_2 \approx \frac{3\,\mathrm{Im}\left\{\sum_{t=0}^{L-1}(L-1-2t)\epsilon_{t,\tau}^*\right\}}{\tau\mu_a^2 L(L^2-1)} \qquad (22)$$

It is then straightforward to show that $E\,\Delta\hat{a}_2 \approx 0$. The second order moment is

$$\begin{aligned} E\,(\Delta\hat{a}_2)^2 &\approx -\frac{9}{4\tau^2\mu_a^4 L^2(L^2-1)^2}\\ &\times \sum_{t=0}^{L-1}\sum_{n=0}^{L-1}(L-1-2t)(L-1-2n) \qquad (23)\\ &\times E\left(\epsilon_{t,\tau}\epsilon_{n,\tau} - 2\epsilon_{t,\tau}\epsilon_{n,\tau}^* + \epsilon_{t,\tau}^*\epsilon_{n,\tau}^*\right) \end{aligned}$$

Under **A2** and **A3** we obtain

$$\begin{aligned} &E\,\epsilon_{t,\tau}\epsilon_{n,\tau} - 2E\,\epsilon_{t,\tau}\epsilon_{n,\tau}^* + E\,\epsilon_{t,\tau}^*\epsilon_{n,\tau}^* = \qquad (24)\\ &-2\sigma_w^2(\sigma_w^2 + 2\sigma_a^2 + 2\mu_a^2)\delta_{t-n} + 2\mu_a^2\sigma_w^2(\delta_{t-n+\tau} + \delta_{t-n-\tau}) \end{aligned}$$

The summations can then be found as

$$\sum_{t=0}^{L-1}\sum_{n=0}^{L-1}(L-1-2t)(L-1-2n)\delta_{t-n} = \frac{L(L^2-1)}{3} \qquad (25)$$

and

$$\sum_{t=0}^{L-1}\sum_{n=0}^{L-1}(L-1-2t)(L-1-2n)\delta_{t-n+\tau} = \frac{L(L^2-1)+2\tau^3+\tau-3L^2\tau}{3}u_{t-2\tau} \quad (26)$$

The summation associated with $\delta_{t-n-\tau}$ is also given by (26) due to symmetry. We then substitute (24) into (23) and use (25) and (26) to obtain the variance expression (14).

6.2. Variance of $\hat{a}_1$

In order to estimate a_1 we consider the demodulated signal,

$$\begin{aligned} X_t^{(1)} &= X_t e^{-j\hat{a}_2 t^2} && (27)\\ &= A_t e^{ja_1 t}e^{-j\Delta\hat{a}_2 t^2} + W_t e^{-j(a_2+\Delta\hat{a}_2)t^2} && (28)\end{aligned}$$

Since the error in $\Delta\hat{a}_2$ is $O(T^{-3})$ we can use the approximation $e^{-j\Delta\hat{a}_2 t^2} \approx 1 - j\Delta\hat{a}_2 t^2$ to obtain

$$X_t^{(1)} \approx A_t e^{ja_1 t}(1-j\Delta\hat{a}_2 t^2)+W_t e^{-ja_2 t^2}(1-j\Delta\hat{a}_2 t^2) \quad (29)$$

The noiseless demodulated signal is $\mu_a s_t^{(1)} = \mu_a s_t e^{-ja_2 t^2} = \mu_a e^{ja_1 t}$. The perturbation in the cyclic mean is then

$$\begin{aligned} \Delta\hat{\mathcal{M}}_{1\mu s^{(1)}} &= \hat{\mathcal{M}}_{1x^{(1)}}(\alpha) - \hat{\mathcal{M}}_{1\mu s^{(1)}}(\alpha) && (30)\\ &= \frac{e^{ja_0}}{T}\sum_{t=0}^{T-1}\Bigg[A_t(1-j\Delta\hat{a}_2 t^2) + \frac{W_t}{s_t^{(1)}}e^{-ja_2 t^2} \\ &\quad \times(1-j\Delta\hat{a}_2 t^2) - \mu_a\Bigg]e^{-j(\alpha-a_1)t} && (31)\\ &= \frac{e^{ja_0}}{T}\sum_{t=0}^{T-1}\rho_t e^{-j(\alpha-a_1)t} && (32)\end{aligned}$$

Using the same first order perturbation method as used for the variance of $\hat{a}_2$, we obtain

$$\Delta\hat{a}_1 \approx \frac{6\,\mathrm{Im}\left\{\sum_{t=0}^{T-1}(T-1-2t)\rho_t^*\right\}}{\mu_a T(T^2-1)} \quad (33)$$

It can be shown without too much difficulty that $E\,\Delta\hat{a}_1 \approx 0$. The second order moment is

$$\begin{aligned} E\,(\Delta\hat{a}_1)^2 &\approx -\frac{9}{\mu_a^2 T^2(T^2-1)^2} \\ &\times \sum_{t=0}^{T-1}\sum_{n=0}^{T-1}(T-1-2t)(T-1-2n) && (34)\\ &\times E\left(\rho_t\rho_n - \rho_n\rho_t^* - \rho_n^*\rho_t + \rho_n^*\rho_t^*\right)\end{aligned}$$

We can evaluate this expression by substituting the approximation of $\Delta\hat{a}_2$, (22), into the expression for ρ_t, (see (31) and (32)). This will result in a large number of terms but the working can be considerably reduced by considering only the highest order terms. Since we are assuming large values of T, this is a valid approximation and will also make it easier to draw meaningful conclusions from the final result. The highest order terms are then

$$\begin{aligned} E\,(\Delta\hat{a}_1)^2 &\approx \frac{9}{\mu_a^2 T^2(T^2-1)^2} && (35)\\ &\times\Bigg\{\frac{9}{4\tau^2\mu_a^2 L^2(L^2-1)^2}\sum_{t=0}^{T-1}\sum_{t=0}^{T-1}\sum_{k=0}^{L-1}\sum_{r=0}^{L-1}t^2 n^2 \\ &\times(T-1-2t)(T-1-2n)(L-1-2k)(L-1-2r) \\ &\times 8\mu_a^2\sigma_w^2(2\mu_a^2\delta_{k-r} - 2\mu_a^2\delta_{k-r+\tau} + \sigma_w^2\delta_{k-r} + 2\sigma_a^2\delta_{k-r}) \\ &+2\sigma_w^4\sum_{t=0}^{T-1}\sum_{n=0}^{T-1}(T-1-2t)(T-1-2n)\delta_{t-n}\Bigg\}\end{aligned}$$

The summations can then be evaluated and after some algebraic manipulation, the variance of $\hat{a}_1$ is found as shown in (15).

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