THE EFFECTS OF SIGNAL-TO-NOISE RATIO MISMATCH ON BAYESIAN MATCHED-FIELD SOURCE LOCALIZATION PERFORMANCE

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ABSTRACT

The signal-to-noise ratio of real data is rarely known with complete certainty. However, Bayesian matched-field processing techniques for ocean acoustic source localization often require the signal-to-noise ratio (SNR) to be known *a priori*. In this paper, the effects of SNR mismatch on the performance of a Bayesian matched-field source localization method, the optimum uncertain field processor [A. M. Richardson and L. W. Nolte, J. Acoust. Soc. Am. **89**(5), 2280-2284 (1991)], are investigated. Theoretical and empirical analyses show that when the maximum *a posteriori* (MAP) estimate is utilized as the source location estimate, the localization performance is unaffected by the uncertainty regarding the SNR, provided that the assumed SNR is sufficiently high.

1. INTRODUCTION

Bayesian approaches to signal detection and parameter estimation often require the signal-to-noise ratio (SNR) to be known *a priori*. Although this requirement is easily satisfied when the algorithms are evaluated through computer simulations, it is nearly impossible to achieve precise knowledge of the SNR in many real life situations, such as experiments conducted in the open ocean, even under controlled conditions. Natural and man-made ambient noise sources, such as seismic disturbances, wind and waves, marine life, and distant shipping traffic, cannot be completely eliminated or exactly known. It is even more difficult to obtain accurate estimates of the SNR in real-time applications, where the source signal is not determined by the experimental protocol and its strength is not precisely known. In practice, the SNR may be estimated in the frequency domain by measuring the received signal not only at the source frequency, but also at frequencies near those which are known to contain the source signal. Presumably these additional measurements contain only noise, and from them an estimate of the noise power, and consequently the SNR, may be obtained. Alternatively, the uncertainties regarding the SNR may be incorporated directly into the processor. In this paper, the effects of SNR mismatch on the performance of a Bayesian matched-field source localization processor which has been developed assuming a known SNR are investigated. Based on the results of the theoretical and empirical analyses, an approach for choosing an assumed SNR when the actual SNR is not known is proposed which mitigates the issue of SNR mismatch.

2. BAYESIAN MATCHED-FIELD SOURCE LOCALIZATION

The optimum uncertain field processor [1] (OUFP) is a Bayesian *a posteriori* probability method used for source localization in an uncertain ocean environment. This technique calculates the *a posteriori* probability of the parameter(s) to be estimated given the received signal, $\mathbf{r}(t)$. The signal emitted by the source is assumed to be a narrowband sinusoid with a known frequency, f_0 [Hz], and the observed time domain signal across the receiving array is assumed to consist of the received source signal, $\mathbf{s}(\mathbf{S},\mathbf{\Psi},\mathbf{\Phi},t)$, plus additive noise, $\mathbf{n}$;

$$\mathbf{r}(\mathbf{S},\mathbf{\Psi},\mathbf{\Phi},t) = \mathbf{s}(\mathbf{S},\mathbf{\Psi},\mathbf{\Phi},t) + \mathbf{n}. \tag{1}$$

The received signal at each array element is a function of the source position relative to the array element, $\mathbf{S}$, the propagation parameters associated with the ocean acoustic waveguide, $\mathbf{\Psi}$, and the amplitude and phase parameters of the source, $\mathbf{\Phi}$.

The frequency transform of the received signal is of the form

$$\mathbf{P}(\mathbf{r}) = A\mathbf{H}(\mathbf{S},\mathbf{\Psi}) + \mathbf{N}, \tag{2}$$

where A is a complex Gaussian random variable with variance σ_A^2 associated with the source parameters $\mathbf{\Phi}$, and $\mathbf{H}(\mathbf{S},\mathbf{\Psi})$ is the replica field for a narrowband source located at the position $\mathbf{S}$ in the ocean $\mathbf{\Psi}$. The observation $\mathbf{P}(\mathbf{r})$ is assumed to contain additive zero-mean Gaussian noise $\mathbf{N}$ with a known spatial covariance matrix $\mathbf{Q}$. In this work, the noise is assumed to be isotropic, consequently $\mathbf{Q} = \sigma_N^2\mathbf{I}$.

Given the assumptions regarding the source amplitude, A, the probability density function for the observation given the source position and the environmental parameters is [1]

$$p(\mathbf{r}|\mathbf{S},\mathbf{\Psi}) = \frac{1}{E(\mathbf{S},\mathbf{\Psi})+1}\exp\left(\frac{\frac{1}{2}|R(\mathbf{r},\mathbf{S},\mathbf{\Psi})|^2}{E(\mathbf{S},\mathbf{\Psi})+1}\right), \tag{3}$$

where $E(\mathbf{S},\mathbf{\Psi}) = \frac{\sigma_A^2}{\sigma_N^2}\mathbf{H}^\dagger(\mathbf{S},\mathbf{\Psi})\mathbf{H}(\mathbf{S},\mathbf{\Psi})$ is related to the energy in the replica field and $R(\mathbf{r},\mathbf{S},\mathbf{\Psi}) = \frac{\sigma_A^2}{\sigma_N^2}\mathbf{H}^\dagger(\mathbf{S},\mathbf{\Psi})\mathbf{P}(\mathbf{r})$ is related to the correlation between the replica field and the observed field. The observed SNR is defined at the receivers and is given by

$$\mathrm{SNR} = \mathcal{E}\left\{E(\mathbf{S},\overline{\mathbf{\Psi}})\right\} = \frac{\sigma_A^2}{\sigma_N^2}\mathcal{E}\left\{\mathbf{H}^\dagger(\mathbf{S},\overline{\mathbf{\Psi}})\mathbf{H}(\mathbf{S},\overline{\mathbf{\Psi}})\right\} \tag{4}$$

where $\mathbf{S}$ is the source position, $\overline{\mathbf{\Psi}}$ is the mean ocean environment, and $\mathcal{E}$ is the expectation operator.

The OUFP calculates $p(\mathbf{S}|\mathbf{r})$, the *a posteriori* probability density function of the source location $\mathbf{S}$ given the received signal $\mathbf{r}$,

$$p(\mathbf{S}|\mathbf{r}) = \frac{\int_{\mathbf{\Psi}} p(\mathbf{r}|\mathbf{S},\mathbf{\Psi})p(\mathbf{S}|\mathbf{\Psi})p(\mathbf{\Psi})d\mathbf{\Psi}}{p(\mathbf{r})}. \tag{5}$$

Given the assumptions regarding the signal model, the probability density function, or ambiguity surface, can be expressed as

$$p(\mathbf{S}|\mathbf{r}) = C(\mathbf{r})p(\mathbf{S})\times \int_{\mathbf{\Psi}} \frac{1}{E(\mathbf{S},\mathbf{\Psi})+1}\exp\left(\frac{\frac{1}{2}|R(\mathbf{r},\mathbf{S},\mathbf{\Psi})|^2}{E(\mathbf{S},\mathbf{\Psi})+1}\right)p(\mathbf{\Psi}|\mathbf{S})d\mathbf{\Psi}, \tag{6}$$

where $C(\mathbf{r})$ is a normalization constant chosen to make $p(\mathbf{S}|\mathbf{r})$ a proper probability density function; $\int_{\mathbf{S}} p(\mathbf{S}|\mathbf{r})d\mathbf{S} = 1$.

When the acoustic environment is known exactly, the environmental integration performed by the OUFP is not necessary, and the matched field processor (MFP) is used to calculate the ambiguity surface;

$$p(\mathbf{S}|\mathbf{r}) = C(\mathbf{r})p(\mathbf{S})\frac{1}{E(\mathbf{S},\mathbf{\Psi})+1}\exp\left(\frac{\frac{1}{2}\left|R(\mathbf{r},\mathbf{S},\mathbf{\Psi})\right|^2}{E(\mathbf{S},\mathbf{\Psi})+1}\right). \tag{7}$$

The MFP provides an upper bound on the attainable level of performance since the environment is known exactly.

3. THEORETICAL ANALYSIS OF BAYESIAN PARAMETER ESTIMATION

Before examining the specific application of Bayesian parameter estimation theory to matched-field source localization, the general approach underlying Bayesian techniques is analyzed in order to gain an intuitive understanding of the role of the SNR in forming the *a posteriori* probability of the parameter given the observed data. The *a posteriori* probability of a desired parameter, θ, given the observed data composed of signal plus noise, $\mathbf{x}(\theta) = \mathbf{s}(\theta)+\mathbf{n}$, computed by Bayesian techniques consists primarily of two components: (1) the information concerning the value taken by the parameter conveyed by the data, represented by $p(\mathbf{x}(\theta)|\theta)$, and (2) the *a priori* knowledge regarding the value of the parameter, represented by $p(\theta)$. This is recognized upon inspection of the expression for the *a posteriori* probability,

$$p(\theta|\mathbf{x}(\theta)) = \frac{p(\mathbf{x}(\theta)|\theta)p(\theta)}{p(\mathbf{x}(\theta))}. \tag{8}$$

In this expression, $p(\mathbf{x}(\theta))$ is a normalizing constant, so the character of the *a posteriori* probability is determined completely by the numerator.

The SNR appears in Eq. 8 upon substitution of the functional forms of the probability density functions. Intuitively, the SNR specifies the relative importance of the observed data and the *a priori* knowledge in forming the *a posteriori* probability density function. For example, if an infinite SNR is assumed, the *a posteriori* probability results solely from the observed data and the *a priori* knowledge is not considered. Conversely, if an SNR of zero ($-\infty$ dB) is assumed, the *a posteriori* probability results solely from the *a priori* knowledge and the observed data is not considered. When an SNR between these two extremes is assumed, the resulting *a posteriori* probability is a combination of the contribution from the observed data and the contribution from the *a priori* knowledge, where the SNR determines relative weighting of these two components. This intuitive result is well-documented in signal detection theory literature [2, 3, 4].

4. THEORETICAL ANALYSIS OF BAYESIAN MATCHED-FIELD SOURCE LOCALIZATION

Now that an intuitive understanding of the role of the SNR in Bayesian approaches has been established, the specific application of Bayesian parameter estimation to matched-field source localization will be examined to determine the effects of SNR mismatch on localization performance. The *a priori* distribution of the source position is normally assumed to be uniform, reflecting the lack of certain prior knowledge regarding the source location. As a result, the characteristics of the *a posteriori* probability are determined by

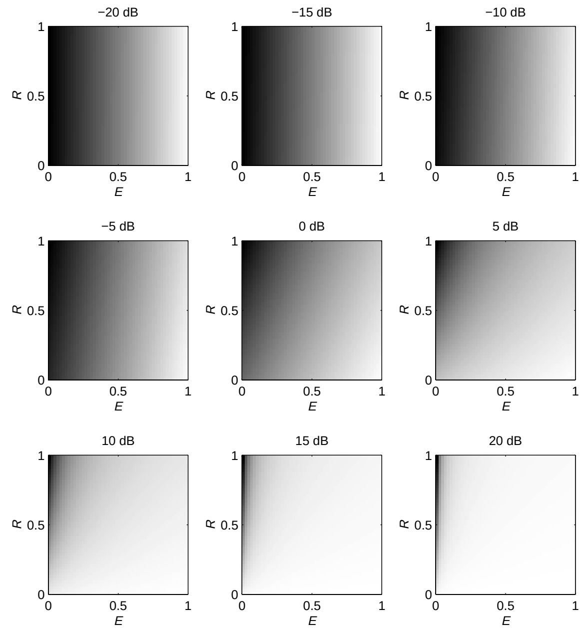

Figure 1: Log probability of source location as a function of $\tilde{E}$ and $\tilde{R}$ for various assumed SNR levels.

the behavior of the *a posteriori* probability of the received signal given the source position and environment, $p(\mathbf{r}|\mathbf{S},\mathbf{\Psi})$. Given the assumptions regarding the signal and noise, the natural log of this probability can be expressed as [1]

$$\ln p(\mathbf{r}|\mathbf{S},\mathbf{\Psi}) = \frac{\frac{1}{2}\mathrm{F}^2\tilde{R}}{\mathrm{F}\tilde{E}+1} - \ln(\mathrm{F}\tilde{E}+1), \tag{9}$$

where $\tilde{E} = \mathbf{H}^\dagger\mathbf{H}$ and $\tilde{R} = \left|\mathbf{H}^\dagger\mathbf{P}\right|^2$. The assumed SNR, denoted by F, is defined as the ratio of the source amplitude variance to the noise variance,

$$\mathrm{F} = \frac{\sigma_A^2}{\sigma_N^2}. \tag{10}$$

This expression reveals the log probability is a function of three terms: $\tilde{R}$, $\tilde{E}$, and F, the *assumed* SNR. To understand the general nature of the probability, it is instructive to examine its behavior as a function of $\tilde{E}$ and $\tilde{R}$ for various levels of assumed SNR, shown in Figure 1. Observe that for low assumed SNR, the log probability is not a function of $\tilde{R}$; it is completely determined by $\tilde{E}$. Moreover, for high assumed SNR, the appearance of the log probability surface does not change as a function of $\tilde{E}$ or $\tilde{R}$. The log probabilities experience a scaling effect as the assumed SNR increases, but the pattern remains the same.

4.1. Approximations to the *A Posteriori* Probability

The preceding observations can be clarified through an asymptotic analysis of the log probability [Eq. (9)]. As F approaches zero, the second term dominates the log probability expression;

$$\lim_{\mathrm{F}\to 0} \ln p(\mathbf{r}|\mathbf{S},\mathbf{\Psi}) = -\ln(\mathrm{F}\tilde{E}+1). \tag{11}$$

As F approaches infinity, the first term is dominant;

$$\lim_{\mathrm{F}\to\infty} \ln p(\mathbf{r}|\mathbf{S},\mathbf{\Psi}) = \frac{\frac{1}{2}\mathrm{F}^2\tilde{R}}{\mathrm{F}\tilde{E}+1}. \tag{12}$$

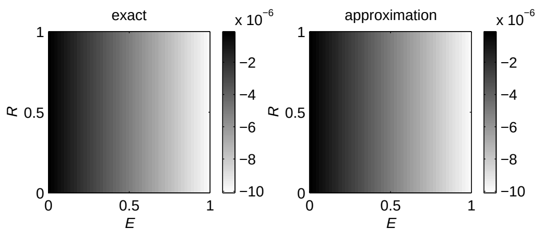

Figure 2: Comparison of exact log probability expression [Eq. (9)] and low SNR approximation [Eq. (11)] as a function of $\tilde{E}$ and $\tilde{R}$ for an assumed SNR of -50 dB.

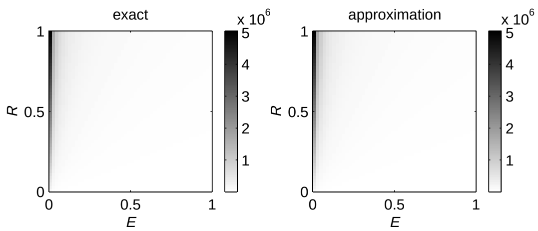

Figure 3: Comparison of exact log probability expression [Eq. (9)] and high SNR approximation [Eq. (12)] as a function of $\tilde{E}$ and $\tilde{R}$ for an assumed SNR of 50 dB.

Based on the discussion on the role of the SNR in Bayesian parameter estimation, these results are intuitively appealing. For low assumed SNR, the log probability depends only on the replica fields, which corresponds to disregarding the observed data and using only the prior knowledge regarding the acoustic propagation to form the probability. For high assumed SNR, the log probability depends only on the correlation between the data and the replica field, corrected for the propagation effects. To confirm the validity of the low and high SNR approximations, the exact and approximate calculations are compared in Figures 2 and 3 for assumed SNR levels of -50 and 50 dB, respectively. As these figures illustrate, the agreement between the exact (P_e) and approximate (P_a) calculations is excellent. A quantitative measure of the agreement is the RMS percent error, defined as

$$\mathrm{RMS}\%\epsilon = \sqrt{\frac{1}{N}\sum_{n=1}^{N}\left(\frac{P_a - P_e}{P_e} \times 100\right)^2}. \qquad (13)$$

The RMS percent error for the high SNR case is 0.21%, and for the low SNR case it is 0.0046%.

4.2. Monotonicity of the *A Posteriori* Probability

The surfaces shown in Figure 1 and the low and high SNR approximations to the log probability expression suggest that for low and high assumed SNR the behavior of the probability (ambiguity) surface is unaffected by the assumed SNR. If the probability is monotonic with respect to the assumed SNR, then the MAP estimate of the source position is unaffected by the assumed SNR. This implies the localization performance is independent of the assumed SNR when the MAP estimate is utilized as the source location estimate. The monotonicity of the probability can be evaluated by calculating its derivative with respect to the assumed SNR, F. When the *a priori* knowledge regarding the source position is uniform, the probability is monotonic if it can be proven that

$$\frac{\partial}{\partial \mathrm{F}}\left\{\int_{\mathbf{\Psi}} p(\mathbf{r}|\mathbf{S},\mathbf{\Psi})\,d\mathbf{\Psi}\right\} \geq 0. \qquad (14)$$

Physical constraints restrict the domain of $\mathbf{\Psi}$, so the limits of integration are bounded. A well-known theorem from real analysis states that if a function and its partial derivative are continuous over the domain of the integrand, then the derivative of the integral and the integral of the derivative are equal [5]. Inspection of the expressions for $p(\mathbf{r}|\mathbf{S},\mathbf{\Psi})$ [Eq. (3)] and its derivative with respect to F,

$$\frac{\partial}{\partial \mathrm{F}} p(\mathbf{r}|\mathbf{S},\mathbf{\Psi}) = \frac{\exp\left(\frac{\frac{1}{2}\mathrm{F}^2\tilde{R}}{\mathrm{F}\tilde{E}+1}\right)}{(\mathrm{F}\tilde{E}+1)^3}\left[\frac{1}{2}\mathrm{F}^2\tilde{E}\tilde{R} + \mathrm{F}(\tilde{R}-\tilde{E}^2) - 1\right], \qquad (15)$$

reveals the conditions necessary to invoke this theorem are satisfied. Hence, the monotonicity test becomes

$$\int_{\mathbf{\Psi}}\left\{\frac{\partial}{\partial \mathrm{F}} p(\mathbf{r}|\mathbf{S},\mathbf{\Psi})\right\} d\mathbf{\Psi} \geq 0. \qquad (16)$$

If each of the terms in the integral is positive, then the integral is also positive, and thus monotonic. In Eq. 15, the first term is always nonnegative, so the monotonicity of this quantity depends on the behavior of the second term. This quadratic equation in F describes a parabola with a minimum at $\mathrm{F} = \frac{\tilde{E}^2-\tilde{R}}{\tilde{E}\tilde{R}}$ and zero crossings at $\mathrm{F} = \frac{\tilde{E}^2-\tilde{R}\pm\sqrt{(\tilde{R}-\tilde{E}^2)^2+2\tilde{E}\tilde{R}}}{\tilde{E}\tilde{R}}$. Therefore, the probability is monotonic with respect to F for sufficiently low or high assumed SNR. Hence, the probability computed by the OUFP is also monotonic, provided that the assumed SNR is chosen such that each of the terms in the integral is monotonic.

Although the low and high SNR levels required to achieve monotonicity are not absolutely defined, this analysis shows the probability is a monotonic function of the assumed SNR for sufficiently low and high assumed SNR. These monotonic regions are where the probability is dominated by either the *a priori* knowledge (low SNR) or the observed data (high SNR). The middle region where the probability is not a monotonic function of the assumed SNR is where the transition from weighting primarily the *a priori* knowledge to considering primarily the observed data occurs.

5. ANALYSIS OF BAYESIAN MATCHED-FIELD SOURCE LOCALIZATION PERFORMANCE

The goal of Bayesian matched-field source localization is to estimate the location of an underwater acoustic source consistently and accurately. A performance metric utilized to quantitatively assess the ability of source localization techniques to correctly identify the source position is probability of correct localization ($\mathrm{P_{CL}}$). The probabilities are calculated through Monte Carlo simulation, and then plotted as a function of SNR to form $\mathrm{P_{CL}}$ curves [6]. In general, the higher the $\mathrm{P_{CL}}$ at a particular SNR, the better the performance of the source localization processor. The effects of SNR mismatch on the performance of Bayesian matched-field source localization are examined through $\mathrm{P_{CL}}$ curves in order to quantify the performance degradation that occurs as a result of SNR mismatch.

The model for the ocean environment utilized in the following simulations, including the environmental parameters and their associated uncertainties, is illustrated in Figure 4. It is an idealized

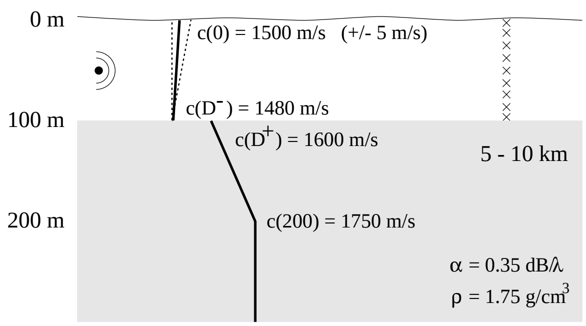

Figure 4: Range independent shallow water ocean model.

range independent shallow water channel similar to an environmental scenario provided for the May 1993 NRL Workshop on Acoustic Models in Signal Processing [7].

A narrowband source at a frequency of 250 Hz is assumed to be located somewhere in the ocean at a horizontal distance of 5 km to 10 km from the vertical receiving array. The array consists of 25 elements with interelement spacing of 4 m, and fully spans the water column. Normal mode theory [8] is utilized to compute the replica fields, and the search area is gridded such that replica fields are calculated every 50 m in range and 2 m in depth.

The overall source localization performance is summarized by $\mathrm{P_{CL}}$ curves, shown in Figures 5 and 6 for the MFP and the OUFP, respectively. Each $\mathrm{P_{CL}}$ curve represents the performance attained for a particular assumed SNR as a function of the actual SNR. The assumed SNR associated with each $\mathrm{P_{CL}}$ curve is displayed to the right of the curve. For comparison purposes, asterisks are plotted for data points where the actual SNR and assumed SNR are equal. An estimated location is considered to be correct if it falls within 100 m (2 gridpoints) of the actual range and 4 m (2 gridpoints) of the actual depth. The results illustrate that for both the MFP and the OUFP, the $\mathrm{P_{CL}}$ is unaffected when the assumed SNR is greater than the true SNR, or high enough to be in the monotonic region. However, the performance is degraded when the assumed SNR is less than the true SNR and not in the monotonic region. These empirical results provide quantitative evidence that when the MAP estimate is utilized as the source location estimate, assuming a high SNR is an appropriate method for managing an unknown SNR.

6. CONCLUSION

The theoretical and empirical findings presented here suggest that when the MFP or OUFP is utilized for source localization and the SNR is not known, as is often the case for real data, a high SNR should be assumed. This approach does not affect the performance level, as indicated by the $\mathrm{P_{CL}}$ curves, and mitigates the problem of SNR mismatch. The interpretation of the ambiguity surfaces as true probabilities is no longer accurate when the SNR is not correctly matched, but the behavior of and features present in the ambiguity surface remain the same and the MAP estimate is not affected. Hence, the performance attained by Bayesian matched-field source localization, as measured by $\mathrm{P_{CL}}$ curves, is not degraded as long as the assumed SNR is greater than the assumed SNR, or high enough to be in the monotonic region. However, when the assumed SNR is less than the true SNR and not in the monotonic region, the performance of Bayesian matched-field source localization suffers.

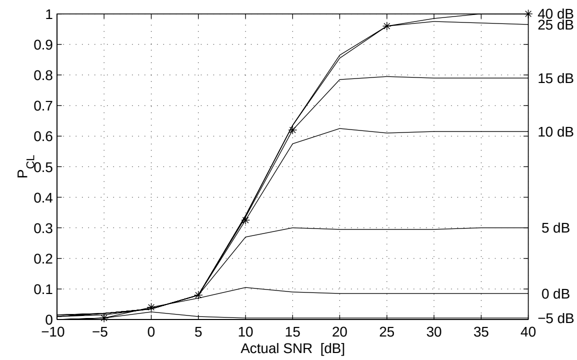

Figure 5: MFP $\mathrm{P_{CL}}$ curves for various assumed SNR levels as a function of the actual SNR.

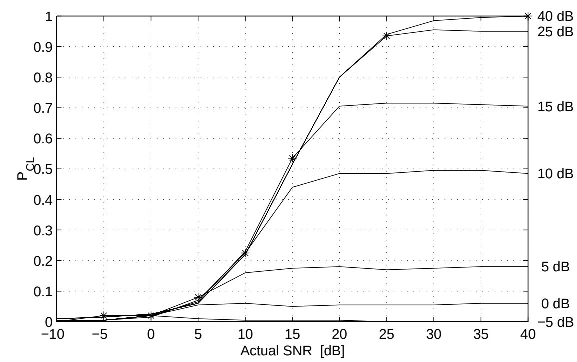

Figure 6: OUFP $\mathrm{P_{CL}}$ curves for various assumed SNR levels as a function of the actual SNR.

7. ACKNOWLEDGMENT

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