IMAGE RECONSTRUCTION WITH TWO-DIMENSIONAL PIECEWISE POLYNOMIAL CONVOLUTION

Stephen E. Reichenbach *Frank Geng*
reich@cse.unl.edu *fgeng@cse.unl.edu*

Department of Computer Science and Engineering
University of Nebraska - Lincoln
Lincoln, Nebraska 68588, USA

ABSTRACT

This paper describes two-dimensional, non-separable, piecewise polynomial convolution for image reconstruction. We investigate a two-parameter kernel with support [-2,2]x[-2,2] and constrained for smooth reconstruction. Performance reconstructing a sampled random Markov field is superior to the traditional one-dimensional cubic convolution algorithm.

I. INTRODUCTION

Image reconstruction is the process of defining a spatially continuous image from a set of discrete samples. It is an important process in image processing theory and is fundamental to many digital image processing operations. Operations such as translation, scaling, rotation, and geometric correction require image values at locations for which no sample is available. The image values at these locations usually are taken to be the convolution of neighboring image samples and a convolution kernel. Common methods for reconstruction include Nearest Neighbor, Linear Interpolation, and Cubic Convolution.

Cubic convolution (CC) has been used for image reconstruction since the 1970's[1]. The usual cubic convolution kernel is a separable, symmetric, piecewise cubic polynomial defined on finite support [-2,2]. Parameteric Cubic Convolution (PCC) is a popular approach in which constraints are imposed to insure continuity and smoothness[4]. Because PCC provides a good compromise between computational complexity and reconstruction accuracy, it is widely used in remote sensing [5]. However, the traditional PCC kernel is a separable product that might not be optimal for two-dimensional images.

In this paper, we investigate the non-separable, symmetric, piecewise polynomial convolution kernel defined on [-2,2] x [-2,2]. In order to insure smooth reconstruction, we impose seven constraints on the polynomial and reduce the number of free coefficients to 2. We then study the role of these two parameters.

The organization of this paper is as follows: Part II formulates the two-dimensional non-separable convolution kernel; Part III gives mathematical analyses of the best value of the two parameters in the convolution kernel; and Part IV is the conclusion and description of future work.

II. TWO-DIMENSIONAL PIECEWISE POLYNOMIAL CONVOLUTION

II. A. Traditional Separable Kernel

The general separable symmetric cubic convolution kernel, which is a polynomial of x and y with degree six, has the form f(x,y)=r(x)r(y), where

$$r(x) = \begin{cases} a_0|x|^3 + b_0|x|^2 + c_0|x| + d_0 & |x| \le 1 \\ a_1|x|^3 + b_1|x|^2 + c_1|x| + d_1 & 1 < |x| \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

To insure smooth, continuous reconstruction and unit response, several constraints are imposed on this kernel:

1. $r(2) = 0$;
2. $r'(2) = 0$;
3. $r(1) = 0$;
4. $r(x)$ is continuous at x=1;
5. $r'(x)$ is continuous at x=1;
6. $r'(x)$ is continuous (thus must be equal to zero) at x=0;
7. $\forall x, \sum_{n=-\infty}^{+\infty} r(x-n) = 1.$

These seven constraints help to reduce the number of coefficients from eight to one.

II. B. Two-dimensional Kernel

We investigate the piecewise polynomial with degree six defined on [-2,2]x[-2,2]. Assuming symmetry about the origin, we consider the four pieces shown in Figure 1 and defined as:

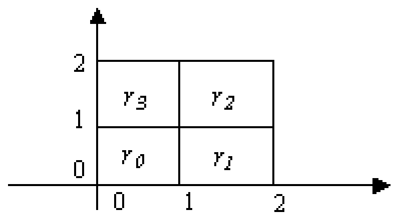

Figure 1. 2-D non-separable convolution kernel defined on [0,2]x[0,2]. This can be generalized to [-2,2]x[-2,2] by symmetry: r(x,-y)=r(x,y) and r(-x,y)=r(x,y).

$$r(x,y)=\begin{cases} r_0(x,y)=\sum_{k=0}^{6}\sum_{i=0}^{k} a_{ki}x^i y^{k-i} & 0\le x,y\le 1 \\ r_1(x,y)=\sum_{k=0}^{6}\sum_{i=0}^{k} b_{ki}x^i y^{k-i} & 1< x\le 2, 0\le y\le 1 \\ r_2(x,y)=\sum_{k=0}^{6}\sum_{i=0}^{k} c_{ki}x^i y^{k-i} & 1< x,y\le 2 \\ r_3(x,y)=\sum_{k=0}^{6}\sum_{i=0}^{k} d_{ki}x^i y^{k-i} & 0\le x\le 1, 1< y\le 2. \end{cases}$$

The total number of free coefficients is 28x4=112. If we assume that r(x,y)=r(y,x), then the number of free coefficients is reduced to 56. The constraints in the one dimensional case are generalized as:

1. $\forall x,\ r(x,2)=0$ and $\forall y,\ r(2,y)=0$;

2. $\forall x,\ \left.\frac{\partial r(x,y)}{\partial y}\right|_{y=2}=0$ and $\forall y,\ \left.\frac{\partial r(x,y)}{\partial x}\right|_{x=2}=0$;

3. $\forall x,\ r(x,1)=0$ and $\forall y,\ r(1,y)=0$;

4. r(x,y) is continuous at r(x,1) and r(1,y);

5. $\forall y, \left.\frac{\partial r(x,y)}{\partial x}\right|_{x=1}$ and $\forall x, \left.\frac{\partial r(x,y)}{\partial y}\right|_{y=1}$ are continuous;

6. $\forall y, \left.\frac{\partial r(x,y)}{\partial x}\right|_{x=0}$ and $\forall x, \left.\frac{\partial r(x,y)}{\partial y}\right|_{y=0}$ are continuous;

7. $\forall x,y\ \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} r(x-m,y-n)=1.$

These seven constraints and the following two propositions reduce the free coefficients from 56 to 2.

Proposition1: For P(x,y), a polynomial of x and y with degree n (n>0), if $\forall y\in R, P(x_0,y)=0$ then P(x,y) must contains a factor $(x-x_0)$.

Proposition2: For $Q(x,y)=(x-x_0)P(x,y)$, both Q(x,y) and P(x,y) polynomials in x and y with degree n (n>0), if $\forall y\in R$, $\left.\frac{\partial Q(x,y)}{\partial x}\right|_{x=x_0}=0$, then Q(x,y) must contains a factor of $(x-x_0)^2$.

According to Propositions 1 and 2 and constraints (1)-(3), r(x,y) can be simplified to:

$$r_0(x,y)=(x-1)(y-1)\sum_{k=0}^{4}\sum_{i=0}^{k} a_{ki}x^i y^{k-i}$$

$$r_1(x,y)=(x-1)(y-1)(x-2)^2\sum_{k=0}^{2}\sum_{i=0}^{k} b_{ki}x^i y^{k-i}$$

$$r_2(x,y)=c(x-1)(y-1)(x-2)^2(y-2)^2$$

$$r_3(x,y)=(x-1)(y-1)(y-2)^2\sum_{k=0}^{2}\sum_{i=0}^{k} d_{ki}x^i y^{k-i}.$$

Further, we impose constraints 4-6 on r(x,y) and after careful calculation[9],

$a_{00}=1$;

$a_{k,0}=a_{k+1,1}$ for k=0,1,2,3 and $a_{40}=0$;

$d_{k,0}=d_{k+1,1}$ for k = 0,1 and $d_{20}=0$;

$d_{10}=0$ and $c=2d_{00}+d_{22}$.

$b_{00}=b_{10}$, $b_{11}=b_{21}$, $b_{22}=0$, $b_{11}=0$

$c=2b_{00}+b_{20}$;

$a_{30}=0$, $b_{00}=a_{20}+2$, $b_{20}=a_{42}+2a_{20}$.

Now, we have only two parameters, a_{42} and a_{20}, and the convolution kernel r(x,y) is:

$$r_0(x,y)=(x-1)(y-1)(1+x+y+a_{20}(x^2+y^2+xy^2+yx^2)+a_{42}x^2y^2$$

$$r_1(x,y)=(x-1)(x-2)^2(y-1)((a_{20}+2)(1+y)+(2a_{20}+a_{42})y^2)$$

$$r_2(x,y)=(4a_{20}+4+a_{42})(x-1)(x-2)^2(y-1)(y-2)^2$$

$$r_3(x,y)=(y-1)(y-2)^2(x-1)((a_{20}+2)(1+x)+(2a_{20}+a_{42})x^2).$$

With $a_{20} = -(\alpha+2)$ and $a_{42} = \beta + (\alpha+2)^2$, the non-separable convolution kernel is r(x,y)=f(x,y)+ β g(x,y), where f(x,y) is the separable convolution kernel:

$$f(x,y) = \begin{cases} (x-1)(y-1)((\alpha+2)x^2 - x - 1) \times \\ \qquad ((\alpha+2)y^2 - y - 1),\ 0 \le x, y \le 1 \\ \alpha(x-1)(x-2)^2(y-1)((\alpha+2)y^2 - y - 1), \\ \qquad 1 < x \le 2,\ 0 \le y \le 1 \\ \alpha(x-1)(y-1)(x-2)^2(y-2)^2,\ 1 < x, y \le 2 \\ \alpha(y-1)(y-2)^2(x-1)((\alpha+2)x^2 - x - 1), \\ \qquad 0 \le x \le 1,\ 1 < y \le 2. \end{cases}$$

and g(x,y) is:

$$g(x,y) = \begin{cases} (x-1)(y-1)x^2y^2 \quad 0 \le x, y \le 1 \\ (x-1)(x-2)^2(y-1)y^2 \quad 1 < x \le 2,\ 0 \le y \le 1 \\ (x-1)(y-1)(x-2)^2(y-2)^2 \quad 1 < x, y \le 2 \\ (y-1)(y-2)^2(x-1)x^2 \quad 0 \le x \le 1,\ 1 < y \le 2. \end{cases}$$

Here, α is the slope of r(x,0) at x=1. When α is fixed, the slope of r(x,x) at x=1 can be related to β.

III. MATHEMATICAL ANALYSIS

To decide the best value for α and β, we consider the Mean Square Error (MSE) caused by image sampling and reconstruction,

$$\varepsilon^2 = \iint_{R^2} \left| f(x,y) - f_r(x,y) \right|^2 dxdy \qquad (1)$$

where f(x,y) is the image before sampling and $f_r(x,y)$ is the reconstructed image.

As in [6], Equation (1) can be expressed as:

$$\varepsilon^2 = \iint_{R^2} \left| \hat{f}(u,v) \right|^2 e^2(u,v) dudv \qquad (2)$$

where $\hat{f}(u,v)$ denotes the Fourier transform of *f(x,y)* and

$$e^2(u,v) = 1 - 2\hat{r}(u,v) + \sum_m \sum_n \left| \hat{r}(u-m, v-n) \right|^2 \qquad (3)$$

The function $e^2(u,v)$ is an image-independent measure of the sampling-reconstruction blur as function of spatial frequency. At any frequency (u,v), the reconstructed fidelity depends on 1- $e^2(u,v)$.

One optimization criteria used in the literature is to minimize the Taylor series of $e^2(u,v)$ at spatial frequency (0,0)[4]. For 2-D piecewise polynomial convolution, the Taylor's expansion of $e^2(u,v)$ at (0,0) is:

$$\begin{aligned} e^2(u,v) = {} & \frac{2\pi^2}{105}(2\alpha+1)^2(u^2+v^2) \\ & + \frac{2\pi^4}{33075}(337 + 1240\alpha + 920\alpha^2 - 416\alpha^3 + 16\alpha^4 \\ & - 216\beta - 416\alpha\beta + 320\alpha^2\beta + 16\beta^2)u^2v^2 \\ & - \frac{\pi^6}{18900}(368 + 769\alpha + 13376\alpha^2)(u^6+v^6) + \ldots \end{aligned} \qquad (4)$$

This implies $\alpha = -1/2$ and $\beta = 0$.

However, real images contain middle and high frequency components (eg. edges and sharp features) which are the most meaningful features in the given image[7]. Thus, the criteria to minimize the MSE at lowest frequency is not appropriate for images that contain middle and high frequency components. As an example, for the scene with a single edge passing through the original at 45°, MSE is minimized with $\alpha = -0.399$ and $\beta = 0.180$. Similarly, for a scene with a square of area s^2 centered at the origin, the MSE optimal values for α and β depend on s, as shown in Figure 2.

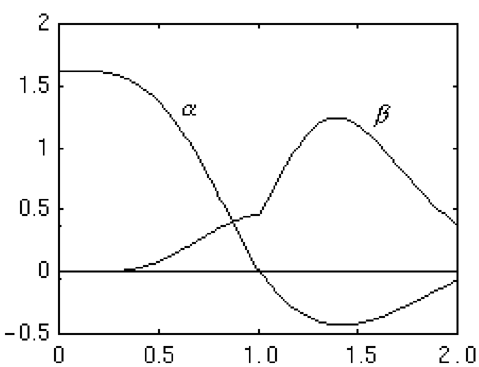

Figure 2. Values of α and β that minimize MSE (Equation 2) for $\hat{f}(u,v) =$ *sinc²(su,sv).*

Most scenes do not contain only a single edge or rectangle. They usually contain more complex structures. Therefore, a more appropriate model for images is the two-dimensional random Markov field, the auto-correlation function of which is:

$$R_{ff}(x,y) = \exp(-\lambda\sqrt{x^2+y^2}). \quad (5)$$

The corresponding power spectrum is

$$E\{\left|\hat{f}(u,v)\right|^2\} = \frac{2\pi\lambda}{(\lambda^2+4\pi^2(u^2+v^2))^{3/2}}. \quad (6)$$

The parameter λ is the expected frequency of edges along any line in the random field[8].

Substituting the expression in (6) into Equation (2) gives an infinite MSE, because the expected sample and reconstruction error is periodic in the spatial domain with a period at [0,1]x[0,1]. Therefore, the reconstruction error in one period is used:

$$\varepsilon^2 = \int_0^1\int_0^1 E\{(f(x,y)-f_r(x,y))^2\}dxdy. \quad (7)$$

With

$$f_r(x,y) = \sum_m\sum_n f(m,n)r(x-m,y-n) \quad (8)$$

Equation (7) is:

$$\begin{aligned}\varepsilon^2 = \int_0^1\int_0^1 & R_{ff}(0,0) - 2\sum_m\sum_n R_{ff}(x-m,y-n)r(x-m,y-n) \\ & + \sum_m\sum_n\sum_p\sum_q R_{ff}(m-p,n-q) \\ & \times r(x-p,y-q)r(x-m,y-n)dxdy. \quad (9)\end{aligned}$$

Solving equations $\frac{\partial\varepsilon^2}{\partial\alpha}=0$ and $\frac{\partial\varepsilon^2}{\partial\beta}=0$, Table 1 lists the best α and β with respect to λ.

λ	α	β
1	0.0207	0.6065
1/2	-0.1622	0.3360
1/4	-0.2271	0.1937
1/8	-0.2512	0.1248
1/16	-0.2611	0.0913
1/32	-0.2654	0.0748

Table 1. Values of α and β that minimize MSE (Equation 2) for a random field model (Equation 5).

IV. CONCLUSION

This paper discusses the two-dimensional, non-separable, piecewise polynomial convolution kernel. Results indicate this kernel is superior to the traditional separable kernel. Further research focuses on relaxing the constraints in order to improve the reconstruction kernel and incorporating restoration.

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