PROCESSING FINITE LENGTH SIGNALS VIA FILTER BANKS WITHOUT BORDER DISTORTIONS: A NON-EXPANSIONIST SOLUTION

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ABSTRACT

In this paper we introduce a novel and general matrix formulation of classical signal extension methods for subband processing of finite length signals. Considering a paraunitary 2-channel filter bank as transformation cell, this new characterization makes it possible to show that perfect reconstruction of finite signals can be ensured without resorting to extra subband samples; thus, by using some traditional signal extension methods, non-expansionist transforms can be defined. Some of these transformations are analyzed to illustrate our theoretical results.

1. INTRODUCTION

It is well known that border distortions appear when reconstructing finite length signals after being analyzed with any paraunitary FIR filterbank. To remove this effect two different approaches have been pointed out by different authors: (a) artificial expansion of the finite signal before the analysis stage [1, 2, 7, 8]; (b) design of border filters or wavelets on the interval [5, 4]. These may be merged into a single approach [6], since the first class of methods lead to the construction of a special type of border filters. Despite these advances on the study of the solutions to this undesired effect, we have observed some unsolved questions:

1. The non-existence of a general formulation for all extension methods; classical extensions such as zero padding, periodic and symmetric extensions seem not to be related to each other.
2. Except for periodic extension and symmetric extension with linear phase filters, all these extensions are expansionist, that is, there are more coefficients in the transform domain than samples in the original signal.

In this paper, we introduce a novel formulation that, in contradiction to the result in [8], proves [3] that not every extension lead to a non-expansionist transform; however, as we will show, there exist some classical extensions that can be re-formulated as non expansionist transforms. So, in addition to the introduction of the formulation, we present the conditions over the transformation matrix to guarantee that, after signal extension, expansion is not necessary.

2. PRELIMINARIES AND NOTATION

Throughout this paper, vectors are denoted by lower case bold letters ($\mathbf{h}$) and matrices by upper case bold letters, fixing their size ($\mathbf{H} = \mathbf{H}_{n\times m}$). The Nth order null and identity matrices are represented by $\mathbf{0}_N$ and $\mathbf{I}_N$ respectively. Let us consider a finite signal $\mathbf{x}$ of even length N, $\mathbf{x} = [x(0), x(1), \cdots, x(N-1)]^T$. We define a extension of $\mathbf{x}$ as $\mathbf{x}_e = [\mathbf{x}_l^T, \mathbf{x}^T, \mathbf{x}_r^T]^T$. This extension is linear if $\mathbf{x}_l$ and $\mathbf{x}_r$ depend linearly on $\mathbf{x}$, that is, if there exist matrices $\mathbf{C}_l, \mathbf{C}_r$ such that $\mathbf{x}_l = \mathbf{C}_l\mathbf{x}$ and $\mathbf{x}_r = \mathbf{C}_r\mathbf{x}$. If both $\mathbf{x}_l$ and $\mathbf{x}_r$ have length M, the extended vector has length $N+2M$ and can be expressed as $\mathbf{x}_e = [\mathbf{C}_l^T, \mathbf{I}_N, \mathbf{C}_r^T]^T\mathbf{x}$.

In addition, we consider the paraunitary filter bank given by the low pass filter $\mathbf{h} = [h(0), h(1), \cdots, h(L-1)]$ (of even length $L > 2$) and the associated high pass filter $\mathbf{g} = [-h(L-1), h(L-2), \cdots, -h(1), h(0)]$. We build the matrix $\mathbf{H}_{m\times(m+L-2)}$, whose m rows contain the filters, adding zeros when necessary. For the sake of simplicity, we will write the $\mathbf{H}_{m\times(m+L-2)}$ block Toeplitz form [5, 6] as:

$$\left[\begin{array}{cccccccc} \mathbf{A}_K & \mathbf{A}_{K-1} & \dots & \mathbf{A}_0 & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_K & \mathbf{A}_{K-1} & \dots & \mathbf{A}_0 & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \ddots & \ddots & \dots & \dots & \ddots & \ddots & \vdots \\ \mathbf{0} & \dots & \dots & \mathbf{0} & \mathbf{A}_K & \mathbf{A}_{K-1} & \dots & \mathbf{A}_0 \end{array}\right],$$

where $\mathbf{0} = \mathbf{0}_{2\times 2}$, $K = L/2 - 1$ and

$$\mathbf{A}_j = \left[\begin{array}{cc} h(2j+1) & h(2j) \\ -h(L-2j-2) & h(L-2j-1) \end{array}\right] \forall j = 0, \dots, K.$$

We set $m = N + 2M - L + 2$ and define the transformation $\mathbf{y}_e = \mathbf{H}_{(N+2M-L+2)\times(N+2M)}\mathbf{x}_e$. This amounts to processing $\mathbf{x}_e$ by means of the analysis filter bank given by $\mathbf{h}$ and $\mathbf{g}$, only retaining the $N+2M-L+2$ central output samples. The whole transformation of the original signal $\mathbf{x}$ can be expressed as $\mathbf{y}_e = \mathbf{G}\mathbf{x}$, where the transformation matrix is

$$\mathbf{G} = \mathbf{H}_{(N+2M-L+2)\times(N+2M)} \left[\begin{array}{c} \mathbf{C}_l \\ \mathbf{I} \\ \mathbf{C}_r \end{array}\right].$$

Let us summarize some properties of the matrix

$$\mathbf{H} \triangleq \mathbf{H}_{(N+2M-L+2)\times(N+2M)}.$$

It has orthonormal rows (not columns), so $\mathbf{H}\mathbf{H}^T = I_{N+2M-2K} \neq \mathbf{H}^T\mathbf{H}$. Nevertheless, if $\mathbf{H}'$ is the matrix containing the $N+2M-4K$ central rows of $\mathbf{H}^T$, we can state that

$$\mathbf{H}'\mathbf{H} = [\mathbf{0}_{(N+2M-4K)\times 2K}\ \mathbf{I}_{N+2M-4K}\ \mathbf{0}_{(N+2M-4K)\times 2K}]. \quad (1)$$

We define $K = L/2 - 1$ and p as the first even number such that $p \geq K$: if K is even, $p = K$ and if K is odd, $p = K + 1$. Let us finally split $\mathbf{H}_{p\times(p+L-2)}$ into three submatrices: $\mathbf{H}_{p\times(p+2K)} = [\mathbf{D}_{p\times K} \quad \mathbf{E}_{p\times p} \quad \mathbf{F}_{p\times K}]$. $\mathbf{D}$ and $\mathbf{F}$ are, respectively, upper and lower block-triangular matrices, and $\mathbf{E}$ is block Toeplitz. They will be useful for describing any matrix $\mathbf{H}_{m\times(m+L-2)}$ (if $m \geq 2p$) in the following way:

$$\left[\begin{array}{ccc} \mathbf{D}_{p\times K} & \mathbf{E}_{p\times p} \quad \mathbf{F}_{p\times K} & \mathbf{0}_{p\times(N-p)} \\ \mathbf{0}_{(m-2p)\times p} & \mathbf{H}_{(m-2p)\times(m+2(K-p))} & \mathbf{0}_{(m-2p)\times p} \\ \mathbf{0}_{p\times(N-p)} & \mathbf{D}_{p\times K} \quad \mathbf{E}_{p\times p} & \mathbf{F}_{p\times K} \end{array}\right]. \qquad (2)$$

3. THE PERFECT RECONSTRUCTION PROBLEM

We are interested in finding which is the minimum number M of extra samples such that we can recover the original signal $\mathbf{x}$ from $\mathbf{y}_e = \mathbf{H}_{(N+2M-2K)\times(N+2M)}\mathbf{x}_e$. If we left multiply $\mathbf{y}_e$ by $\mathbf{H}'$ and remind (1), we obtain $\mathbf{H}'\mathbf{y}_e = \mathbf{H}'\mathbf{H}\mathbf{x}_e = \mathbf{x}_P$, where $\mathbf{x}_P$ contains the $N+2M-4K$ central components of $\mathbf{x}_e$. This means that we can perfectly reconstruct at least the $N+2M-4K$ central components of $\mathbf{x}$.

Then, the first idea is to take $M = 2K$, so that we can perfectly reconstruct the whole vector $\mathbf{x} = \mathbf{x}_P$. This leads to a length $N+4K$ extended signal $\mathbf{x}_e$, which, after being analyzed, results in a $N+2K$ length vector $\mathbf{y}_e$. So we have to work with $N+2K$ subband samples, that is, we are dealing with an expansionist transform. In other words, by applying a length L 2-channel paraunitary filter bank to a finite signal, we would need to construct $\mathbf{x}_e$ by adding $L-2$ extra samples per border to achieve perfect reconstruction. We will show that for many kinds of extensions, it is possible to reduce this quantity to $K = L/2 - 1$ extra samples and, consequently, to work with non-expansionist transforms.

Let us take $M = K$ and consider an arbitrary linear extension of $\mathbf{x}$, $\mathbf{x}_e$. The transform vector $\mathbf{y}_e = \mathbf{H}_{N\times(N+2K)}\mathbf{x}_e = \mathbf{G}\mathbf{x}$ and $\mathbf{x}$ have the same length N, so $\mathbf{x}$ can be perfectly reconstructed from $\mathbf{y}_e$ if and only if the square transformation matrix $\mathbf{G}$ is invertible. Under this condition, the transformation is non-expansionist. *Matrices* $\mathbf{G}$ *represent the transformations of every linear extension of a finite signal based on* $\mathbf{H}_{N\times(N+2K)}$. Next we find its general expression.

If $N \geq 2K$, we denote $\mathbf{x} = [\mathbf{x}_a^T \ \mathbf{x}_c^T \ \mathbf{x}_b^T]^T$, where $\mathbf{x}_a$ and $\mathbf{x}_b$ contain, respectively, the first and last K components of $\mathbf{x}$, and $\mathbf{x}_c$ the remaining central ones. Let us also write $\mathbf{C}_l = [\mathbf{C}^{l,a}\mathbf{C}^{l,c}\mathbf{C}^{l,b}]$ and $\mathbf{C}_r = [\mathbf{C}^{r,a}\mathbf{C}^{r,c}\mathbf{C}^{r,b}]$, where $\mathbf{C}^{l,a}$, $\mathbf{C}^{l,b}$, $\mathbf{C}^{r,a}$, $\mathbf{C}^{r,b}$ are square submatrices of K order. Left and right extensions can now be written as $\mathbf{x}_l = \mathbf{C}^{l,a}\mathbf{x}_a + \mathbf{C}^{l,c}\mathbf{x}_c + \mathbf{C}^{l,b}\mathbf{x}_b$ and $\mathbf{x}_r = \mathbf{C}^{r,a}\mathbf{x}_a + \mathbf{C}^{r,c}\mathbf{x}_c + \mathbf{C}^{r,b}\mathbf{x}_b$.

Now, using (2) for $m = N$, we perform the block product

$$\mathbf{G} = \mathbf{H}_{N\times(N+2K)} \left[\begin{array}{ccc} \mathbf{C}^{l,a} & \mathbf{C}^{l,c} & \mathbf{C}^{l,b} \\ \mathbf{I}_K & \mathbf{0}_{K\times(N-2K)} & \mathbf{0}_K \\ \mathbf{0}_K & \mathbf{I}_{N-2K} & \mathbf{0}_K \\ \mathbf{0}_K & \mathbf{0}_{K\times(N-2K)} & \mathbf{I}_K \\ \mathbf{C}^{r,a} & \mathbf{C}^{r,c} & \mathbf{C}^{r,b} \end{array}\right];$$

if K is even, whenever $N \geq 3K$, we obtain

$$\mathbf{G} = \left[\begin{array}{ccc} \mathbf{D}\mathbf{C}^{l,a} + \mathbf{E} & \mathbf{D}\mathbf{C}^{l,c} + [\mathbf{F}\ \mathbf{0}_{K\times(N-3K)}] & \mathbf{D}\mathbf{C}^{l,b} \\ & \mathbf{H}_{(N-2K)\times N} & \\ \mathbf{F}\mathbf{C}^{r,a} & \mathbf{F}\mathbf{C}^{r,c} + [\mathbf{0}_{K\times(N-3K)}\ \mathbf{D}] & \mathbf{E} + \mathbf{F}\mathbf{C}^{r,b} \end{array}\right].$$

We will show that that invertibility of $\mathbf{G}$ is independent of $\mathbf{C}^{l,c}$ and $\mathbf{C}^{r,c}$; so we can consider them as null matrices. In other words, we will assume that the left and right extensions of $\mathbf{x}_e$ depend on the initial and final portions of the original signal. In this way, we get the simplest general expression of $\mathbf{G}$, for all K whenever $N \geq 3K$:

$$\mathbf{G} = \left[\begin{array}{c} [\mathbf{D}\mathbf{C}^{l,a}\ \mathbf{0}_{p\times(p-K)}] + \mathbf{E}\ \mathbf{F}\ \mathbf{0}_{p\times(N-2K-p)}\ \mathbf{D}\mathbf{C}^{l,b} \\ \mathbf{0}_{(N-2p)\times(p-K)}\ \mathbf{H}_{(N-2p)\times N}\ \mathbf{0}_{(N-2p)\times(p-K)} \\ \mathbf{F}\mathbf{C}^{r,a}\ \mathbf{0}_{p\times(N-2K-p)}\ \mathbf{D}\ \mathbf{E} + [\mathbf{0}_{p\times(p-K)}\ \mathbf{F}\mathbf{C}^{r,b}] \end{array}\right]. \qquad (3)$$

3.1. Classification of Linear Extensions of Finite Signals

In this section we will study all kinds of linear extensions with their associate matrices $\mathbf{G}$. We classify them in *circular* and *non-circular extensions*. We say that a extension is non-circular when the extra samples added at each border of the original signal depend only on the K samples of the *same* border. That is, when $\mathbf{C}^{l,b}$ and $\mathbf{C}^{r,a}$ are null matrices. The remaining ones are circular extensions. Among non-circular extensions, we consider predictive and non-predictive ones.

3.1.1. Predictive Non-circular Extensions

For this kind of extensions, extra samples are recursively defined as *a fixed linear combination* of the original signal. That is, there exist linear prediction coefficients $c_1, c_2, \ldots, c_K$ related to x_a, and each new sample of x_l is built from the K samples on its right through the combination: $x_l(j) = \sum_{n=1}^{K-j-1} c_n x_l(j+n) + \sum_{n=0}^{j} c_{K-j+n} x_a(n), \quad j = 0, \ldots, K-1$. In the same way, from x_b, we define each sample of x_r by linear prediction over its left K samples, with coefficients $c'_K, c'_{(K-1)}, \ldots, c'_1$: for every $j = 0, \ldots, K-1$,

$$x_r(j) = \sum_{n=0}^{K-j-1} c'_{K-n} x_b(j+n) + \sum_{n=0}^{j-1} c'_{j-n} x_r(n).$$

This is a linear extension since it can be easily shown that $\mathbf{x}_l = \mathbf{C}^K\mathbf{x}_a$ and $\mathbf{x}_r = \mathbf{C}'^K\mathbf{x}_b$, where $\mathbf{C}$ and $\mathbf{C}'$ are the Frobenius matrices:

$$\mathbf{C} = \left[\begin{array}{cc} c_1 \ldots & \ldots c_K \\ \mathbf{I}_{K-1} & \mathbf{0}_{(K-1)\times 1} \end{array}\right], \ \mathbf{C}' = \left[\begin{array}{cc} \mathbf{0}_{(K-1)\times 1} & \mathbf{I}_{K-1} \\ c'_K \ldots & \ldots c'_1 \end{array}\right].$$

Finally, we substitute in (3) and obtain the matrix $\mathbf{G}_l$ associated to the predictive linear extension; now we write it for even K (hence $p = K$):

$$\mathbf{G}_l = \left[\begin{array}{c} \mathbf{D}\mathbf{C}^K + \mathbf{E} \quad \mathbf{F} \quad \mathbf{0}_{K\times(N-2K)} \\ \mathbf{H}_{(N-2K)\times N} \\ \mathbf{0}_{K\times(N-2K)} \quad \mathbf{D} \quad \mathbf{E} + \mathbf{F}\mathbf{C}'^K \end{array}\right].$$

To conclude this subsection we will analyze two interesting examples of this kind of extensions: the classical zero padding and polynomial extension.

3.1.2. Zero Padding

It consists of adding null samples at each border of signal $\mathbf{x}$; that is, $\mathbf{x}_l$ and $\mathbf{x}_r$ are null vectors. It can be considered as a predictive extension whose prediction coefficients are all equal to zero. In this case, $\mathbf{C}_l = \mathbf{C}_r = \mathbf{0}_{2K\times N}$ and the associated transformation matrix $\mathbf{G}_0$ contains only the N central columns of $\mathbf{H}_{N\times(N+2K)}$. It is also a block Toeplitz matrix; if K is even it adopts the following expression:

$$\begin{bmatrix} \mathbf{A}_{K/2} & \mathbf{A}_{K/2-1} & \dots & \mathbf{A}_0 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{A}_{K/2+1} & \mathbf{A}_{K/2} & \mathbf{A}_{K/2-1} & \dots & \mathbf{A}_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \dots & \dots & \ddots & \vdots \\ \mathbf{A}_K & \mathbf{A}_{K-1} & \dots & \dots & \dots & \mathbf{A}_0 & \ddots \\ \mathbf{0} & \mathbf{A}_K & \ddots & \dots & \dots & \dots & \ddots \\ \vdots & \ddots & \ddots & \dots & \dots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{A}_K & \dots & \dots & \mathbf{A}_{K/2} \end{bmatrix},$$

and if K is odd, its Toeplitz blocks are mixed versions of these ones. Let us finally remind that the main drawback of zero padding is the generation of artificial high frequencies in the transform vector. A solution to this problem is to consider "smooth" extensions such as that presented in the following section.

3.1.3. Polynomial Extension

Algebraic manipulations guarantee the existence and uniqueness of a degree $d < K$ polynomial that passes through the K samples of $\mathbf{x}_a$. Hence, we set $\mathbf{x}_l$ as the values taken by this polynomial on the K points on the left of $\mathbf{x}$; similarly, we construct $\mathbf{x}_r$ from the values of $\mathbf{x}_b$. Now $\mathbf{x}_e$ is the polynomial extension of $\mathbf{x}$. We can see, using the theory of finite differences, that this is in fact another example of linear predictive extension, whose coefficients are

$$c_j = c_j' = \begin{cases} (-1)^{(j+1)}\binom{d+1}{j} & \text{if} \quad 1 \le j \le d+1 \\ 0 & \text{if} \quad d+2 \le j \le K; \end{cases}$$

so we can write the transformation matrix associated to this extension as a particular case of $\mathbf{G}_l$.

Going back to the smoothness concept, we have to remind that this depends on the number n of vanishing moments of $\mathbf{h}$. In fact, transform vectors of polynomials up to degree $n-1$ do not present high frequencies components: they are considered smooth functions. Hence, after polynomial extension (of degree $d \le K-1$), any signal will keep its smoothness whenever $n \ge K$. For instance, length L Daubechies filters have $n = L/2 = K+1$ vanishing moments; this means that with these filters, polynomial extension is always a smooth extension.

3.1.4. Non-predictive Non-circular Linear Extensions

As an outstanding example of infinite non-circular linear extensions we must analyze the classical symmetric extension. In this case $\mathbf{x}_l = \mathbf{P}_K\mathbf{x}_a$ and $\mathbf{x}_r = \mathbf{P}_K\mathbf{x}_b$, where $\mathbf{P}_K$ is the permutation matrix with 1's on the antidiagonal. The transformation matrix associated to symmetric extension is, for even K,

$$\mathbf{G}_s = \begin{bmatrix} \mathbf{DP}_K + \mathbf{E} \quad \mathbf{F} \quad \mathbf{0}_{K\times(N-2K)} \\ \mathbf{H}_{(N-2K)\times N} \\ \mathbf{0}_{K\times(N-2K)} \quad \mathbf{D} \quad \mathbf{E}+\mathbf{FP}_K \end{bmatrix}.$$

3.1.5. Circular Extensions

In this group we find all kind of extensions whenever $\mathbf{C}^{l,a}$ or $\mathbf{C}^{r,b}$ are not null matrices. We consider two examples; the first one leads to the famous periodic extension.

- *Periodic extension:* in this case we take $\mathbf{x}_l = \mathbf{x}_b$ and $\mathbf{x}_r = \mathbf{x}_a$. Hence, $\mathbf{C}^{l,b} = \mathbf{C}^{r,a} = \mathbf{I}_K$ and $\mathbf{C}^{l,a} = \mathbf{C}^{r,b} = \mathbf{0}_K$. The associated matrix $\mathbf{G}_{per}$ is, for any K:

$$\begin{bmatrix} \mathbf{E} \quad \mathbf{F} \quad \mathbf{0}_{p\times(N-2K-p)} \quad \mathbf{D} \\ \mathbf{0}_{(N-2p)\times(p-K)} \quad \mathbf{H}_{(N-2p)\times N} \quad \mathbf{0}_{(N-2p)\times(p-K)} \\ \mathbf{F} \quad \mathbf{0}_{p\times(N-2K-p)} \quad \mathbf{D} \quad \mathbf{E} \end{bmatrix}.$$

  It is a block circulant orthogonal matrix, because its rows contains the even shifts of the orthogonal filters. Hence, the transformation is always invertible; however this extension may lead again to artificial high frequencies.

- *Smooth circular extension:* If we desire the periodization process not to introduce discontinuities in $\mathbf{x}_e$, we can extend the signal before periodization in the following way: we would like to define $\mathbf{x}_r,\ \mathbf{x}_l$ so that the signal $[\mathbf{x}_b^T,\ \mathbf{x}_r^T,\ \mathbf{x}_l^T,\ \mathbf{x}_a^T]$ is smooth enough. From the last samples of $\mathbf{x}_b$ and the first ones of $\mathbf{x}_a$, we can construct it through polynomial interpolation. And this is a linear process which involves both $\mathbf{x}_a$ and $\mathbf{x}_b$ for each border, that is, a circular linear extension. If we are not going to periodize afterwards, it is better to apply the matrix $\mathbf{G}$ associated to polynomial extension directly.

3.2. Invertibility of the Transformation

Considering the cases analyzed in the previous section, we can only guarantee the invertibility of the orthogonal matrix associated to periodic extension. Next we give necessary and sufficient conditions for $\mathbf{G}$ to have an inverse. It is equivalent to be able to perfectly reconstruct $\mathbf{x}$ from $\mathbf{y}_e = \mathbf{Gx}$. We remind that left multiplying $\mathbf{y}_e$ by $\mathbf{H}'$, we obtain the $N-2K$ central samples of $\mathbf{x}$, that is, $\mathbf{x}_c$. In order to determine $\mathbf{x}_a,\ \mathbf{x}_b$ from $\mathbf{G}[\mathbf{x}_a^T,\ \mathbf{x}_c^T,\ \mathbf{x}_b^T]^T = \mathbf{y}_e$, we rearrange this linear system by moving the known vector $\mathbf{x}_c$ to the right side, and using (3) it becomes

$$\mathbf{S}\begin{bmatrix} \mathbf{x}_a \\ \mathbf{x}_b \end{bmatrix} = \begin{bmatrix} \mathbf{DC}^{l,a} + \mathbf{E}* & \mathbf{DC}^{l,b} \\ \mathbf{0}_{p\times(p-K)}\ \mathbf{D}* & \mathbf{0}_{p\times K} \\ \mathbf{0}_{p\times K} & *\mathbf{F}\ \mathbf{0}_{p\times(p-K)} \\ \mathbf{FC}^{r,a} & *\mathbf{E} + \mathbf{FC}^{r,b} \end{bmatrix} \begin{bmatrix} \mathbf{x}_a \\ \mathbf{x}_b \end{bmatrix} = \mathbf{y}'.$$

In this new linear system we have denoted as $\mathbf{A}*$ (respectively $*\mathbf{A}$) the submatrix of $\mathbf{A}$ constructed from its first (respectively last) K columns. If K is even, then the asterisk may be omitted. Independently of $\mathbf{y}'$, the solution is unique if and only if the columns of the system matrix $\mathbf{S}$ are linearly independent. In that case, multiplying by the pseudoinverse gives back $\mathbf{x}_a,\ \mathbf{x}_b$. We have obtained the following result:

Proposition 1 $\mathbf{G}$ *has an inverse if and only if its submatrix* $\mathbf{S}$ *has maximum rank* $(2K)$.

Thus, we have given a characterization of regular matrices $\mathbf{G}$. Besides, this condition is independent of the extension matrices $\mathbf{C}^{l,c},\ \mathbf{C}^{r,c}$. Moreover, under this assumption, we have described a practical synthesis algorithm for $\mathbf{x}$ from $\mathbf{y}_e$. In our work, we have also found other characterizations, for instance:

Proposition 2 $\mathbf{G}$ *is regular if and only if there exists any solution* $\mathbf{X}$ *to the following matrix equation:*

$$\mathbf{SX} = \left[\begin{array}{cc} \mathbf{D} & \mathbf{0}_{p\times K} \\ \mathbf{0}_{2p\times K} & \mathbf{0}_{2p\times K} \\ \mathbf{0}_{p\times K} & \mathbf{F} \end{array} \right].$$

4. EXAMPLE

We have considered Daubechies filters of length 10, and have built matrices $\mathbf{G}$ corresponding to the polynomial and periodic extension. Figure 1 shows the three subband of a cubic finite signal: smooth polynomial extension (Figure 1(a)) presents the best performance on the subbands; on the other hand, periodic extension (Figure 1(b)) introduces discontinuities which create spurious frequencies in every subband; this is important when studying the effects of quantization errors. Figure 2 shows reconstruction errors for these signals after multiplying by the corresponding inverse matrix , (b) and (d), or using the reconstruction algorithm proposed in the previous section, (a) and (c). We have also tested Daubechies filters up to length 34: we conclude that such inverses exist for all the types of extensions studied in this paper.

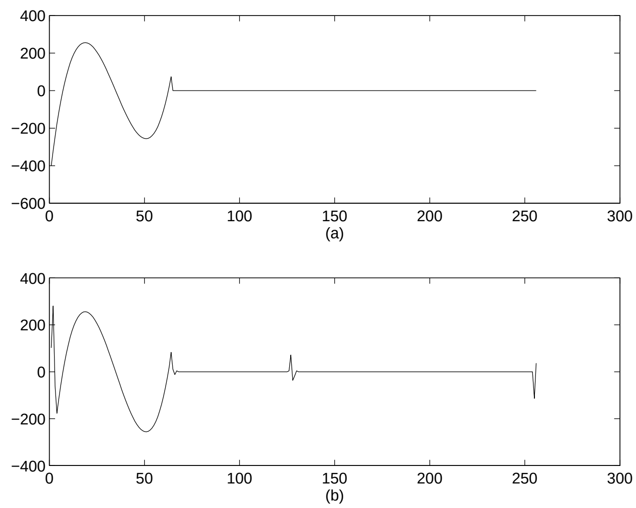

Figure 1: Subband transforms of a cubic signal after: (a) polynomial extension and (b) periodic extension.

5. CONCLUSIONS

We have introduced the general formulation of the subband processing of an extended finite length signal. New conditions for non-expansionist invertible transforms have been given, and practical examples were shown. Current research is being oriented to the general demonstration of invertibility of each one of those transforms regardless the filter bank, and the design of new subband transforms of finite signals with improved properties. Further work can also deal with the study of the effects of quantization errors on the reconstruction of the finite signal.

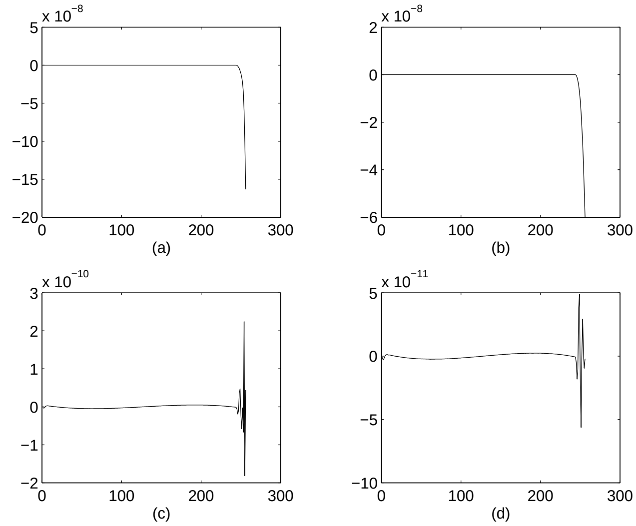

Figure 2: Signal reconstruction errors for the cubic finite signal using: (a) polynomial extension and proposed reconstruction algorithm; (b) polynomial extension and multiplication by inverse matrix; (c) periodization and proposed reconstruction algorithm and (d) periodization and multiplication by inverse matrix.

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