

NEW FAST ALGORITHMS OF MULTIDIMENSIONAL FOURIER AND RADON DISCRETE TRANSFORMS

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ABSTRACT

This paper describes a fast new n -D Discrete Radon Transform (DRT) and a fast exact inversion algorithm for it, without interpolating from polar to Cartesian coordinates or using the backprojection operator. New approach is based on the fast Nussbaumer's Polynomial Transform (NPT).

1. INTRODUCTION

The Radon Transform (RT) and its ill-conditioned inverse were first formulated by J. Radon in 1917. Currently, the RT is used in a wide variety of applications including tomography, ultrasound, optics, and geophysics, to name a few.

Discrete versions of the classical RT are being used in signal processing and there is an extensive literature devoted to this subject. Procedures which are discrete versions of the RT are known as slant stack [1], τ -P transform [2], [3], velocity filtering [4]. These procedures are successfully used in various applications. The fast discrete 2-D Radon transform algorithms are presented in [6],[7]. In the last paper fast algorithms is presented only for the direct 2-D transform.

In this paper, we introduce new direct and inverse DRT and show that they admit fast computation by the fast Nussbaumer Polynomial Transform (NPT) [5].

2. DISCRETE FOURIER AND RADON TRANSFORMS

Let $\mathcal{D}^\nu(N) = Z_N \mathbf{e}_1 + Z_N \mathbf{e}_2 + \dots + Z_N \mathbf{e}_\nu$ be ν -D a discrete parallelepiped. Its elements are column vectors $\mathbf{i} := (i_1, \dots, i_\nu)^* = |\mathbf{i}\rangle$, where $i_1, \dots, i_\nu \in Z_N$. Let $\mathcal{D}^{*\nu}(N)$ be a dual parallelepiped consisting of vector-rows $\mathbf{k} := (k_1, k_2, \dots, k_\nu) = \langle \mathbf{k}|, k_1, \dots, k_\nu \in Z_N$.

Definition 1 The unitary operators \mathcal{F}_ν and \mathcal{F}_ν^{-1} acting by rules

$$\mathcal{F}\{f(\mathbf{i})\} := \frac{1}{\sqrt{N}} \sum_{\mathbf{i} \in \mathcal{D}^\nu} f(\mathbf{i}) e^{-\frac{j2\pi}{N} \langle \mathbf{i}|\mathbf{k}\rangle} = F(\mathbf{k}),$$

$$\mathcal{F}_\nu^{-1}\{F(\mathbf{k})\} := \frac{1}{\sqrt{N}} \sum_{\mathbf{k} \in \mathcal{D}_N^{*\nu}} F(\mathbf{k}) e^{\frac{j2\pi}{N} \langle \mathbf{i}|\mathbf{k}\rangle} = f(\mathbf{i})$$

are called *direct and inverse discrete ν -D Fourier transforms* (DFT), where $\langle \mathbf{i}|\mathbf{k}\rangle := \sum_{s=1}^\nu i_s k_s$.

For convenience we omit in the following the normalization factor (constant) $1/\sqrt{N}$.

Let $\{\alpha^\circ\} \in \mathcal{D}^{*\nu}(N)$ be minimal such vector set that the rays $\{a\alpha^\circ \mid a = 1, 2, \dots, N-1\}$ cover the whole parallelepiped $\mathcal{D}^*(N) : \{a\alpha^\circ \mid a = 1, 2, \dots, N-1\}$. Then we can write that $F(a\alpha^\circ) =$

$$= \sum_{\mathbf{i} \in \mathcal{D}^\nu} f(\mathbf{i}) e^{-\frac{j2\pi a}{N} \langle \alpha^\circ|\mathbf{i}\rangle} = \sum_{p=0}^{q-1} \left(\sum_{\langle \alpha^\circ|\mathbf{i}\rangle = p} f(\mathbf{i}) \right) e^{\frac{2j\pi a p}{N}},$$

or

$$F(a\alpha^\circ) = \sum_{p=0}^{q-1} \hat{f}(\alpha^\circ, p) e^{\frac{2j\pi a p}{N}}, \quad (1)$$

where

$$\hat{f}(\alpha^\circ, p) := \mathcal{R}_\nu\{f(\mathbf{i})\} = \sum_{\langle \alpha^\circ|\mathbf{i}\rangle = p} f(\mathbf{i}).$$

Definition 2 The function $\hat{f}(\alpha^\circ, p)$ which is equal to the sum of values of the signal $f(\mathbf{i})$ on the discrete hyperplane $\langle \alpha^\circ|\mathbf{i}\rangle = p$ is called *Discrete Radon Transform* (DRT) of $f(\mathbf{i})$ [6].

The expression (1) means that ν -D DFT \mathcal{F}_ν is a composition of DRT \mathcal{R}_ν and a set 1-D DFT's. The total number of 1-D DFT is equal to the power of the set $\{\alpha^\circ\}$. Every 1-D DFT acts along the ray $\{a\alpha^\circ \mid a =$

$1, 2, \dots, L(\alpha^\circ)\}$, where $L(\alpha^\circ)$ is the length of the ray. It is necessary to find such $\{\alpha^\circ\}$ that would give DRT with minimum computational complexity. Note that the classical "rown/column separable" ν -D DFT is reduced to $\nu N^{\nu-1}$ 1-D DFT's of the length N .

Theorem 1 If $N = q$ is a prime integer then the total number of rays $\text{RAY}(\nu, q)$ that cover $\mathcal{D}^{*\nu}(q)$ is equal to

$$\text{RAY}(\nu, q) = (q^\nu - 1)/(q - 1),$$

and each of rays has length $L(n, q) = q$. All rays spanned by the following vectors of set $\{\alpha^\circ\}$:

- $\{\alpha^\circ\}^\nu := (k_1, \dots, k_{\nu-2}, k_{\nu-1}, 1),$
- $\{\alpha^\circ\}^{\nu-1} := (k_1, \dots, k_{\nu-2}, 1, 0), \dots, \dots, \dots,$
- $\{\alpha^\circ\}^2 := (k_1, 1, 0, \dots, 0),$
- $\{\alpha^\circ\}^1 := (1, 0, \dots, 0),$

where $k_i \in \mathbf{Z}_q, i = 1, 2, \dots, n$.

3. FAST ν -D RADON AND FOURIER TRANSFORMS

For ν -D DFT we have

$$F(k_1, k_2, \dots, k_\nu) = \sum_{i_1=0}^{q-1} \sum_{i_2=0}^{q-1} \dots \sum_{i_\nu=0}^{q-1} f(i_1, i_2, \dots, i_\nu) W^{k_1 i_1 + \dots + k_\nu i_\nu},$$

and the set $\{\alpha^\circ\} = \{\alpha^\circ\}^\nu \cup \{\alpha^\circ\}^{\nu-1} \cup \dots \cup \{\alpha^\circ\}^2 \cup \{\alpha^\circ\}^1$ of $(q^\nu - 1)/(q - 1)$ vectors. How fast can we fast calculate DRT, i.e. the following sums $\hat{f}(p, \alpha^\circ) =$

$$\left\{ \begin{array}{l} \sum_{k_1 i_1 + k_2 i_2 + \dots + k_{\nu-1} i_{\nu-1} + k_\nu = p} f(i_1, \dots, i_{\nu-1}, i_\nu), \\ \sum_{k_1 i_1 + k_2 i_2 + \dots + k_{\nu-2} i_{\nu-2} + k_{\nu-1} = p} f(i_1, \dots, i_{\nu-1}, i_\nu), \\ \dots \\ \sum_{k_1 i_1 + k_2 = p} f(i_1, i_2, \dots, i_{\nu-1}, i_\nu), \\ \sum_{i_1 = p} f(i_1, i_2, \dots, i_{\nu-1}, i_\nu)? \end{array} \right. \quad (2)$$

Step 1.

1. For clarify we introduce following notation

$$f_\nu(i_1, i_2, \dots, i_\nu) := f(i_1, i_2, \dots, i_\nu).$$

2. For fast calculation of sums (2) we *interpretate* the ν -D scalar-valued signal $f_\nu(i_1, i_2, \dots, i_\nu)$ as $(\nu-1)$ -D polynomial-valued signal: $\mathcal{F}_\nu(z)(i_1, i_2, \dots, i_{\nu-1}) =$

$$= \sum_{i_\nu=0}^{q-1} f_\nu(i_1, i_2, \dots, i_\nu) z^{i_\nu} \pmod{(z^q - 1)} \quad (3)$$

that has polynomial values. The space of these signals will be denoted as $\mathbf{L}(\mathbf{Z}_q^{\nu-1}, \mathbf{R}[z]/(z^q - 1))$. In this space we introduce according to Nussbaumer the polynomial-valued basis

$$\mathcal{E}(z)_{k_1, \dots, k_{\nu-1}}(i_1, \dots, i_{\nu-1}) = z^{k_1 i_1 + \dots + k_{\nu-1} i_{\nu-1}},$$

where $k_1, i_1, \dots, k_{\nu-1}, i_{\nu-1} \in \mathbf{Z}_q$.

3. Let us find the polynomial-valued spectrum

$$\begin{aligned} \hat{\mathcal{F}}_\nu(z)(k_1, \dots, k_{\nu-1}) &= \\ &= \sum_{i_1=0}^{q-1} \dots \sum_{i_{\nu-1}=0}^{q-1} \mathcal{F}_{\nu-1}(z)(i_1, \dots, i_{\nu-1}) z^{k_1 i_1 + \dots + k_{\nu-1} i_{\nu-1}} \end{aligned} \quad (4)$$

In operator notation this transform can be describe as

$$\begin{aligned} \hat{\mathcal{F}}_\nu(z)(k_1, \dots, k_{\nu-1}) &= \\ &= \left(\bigotimes_{s=1}^{\nu-1} \mathcal{N}_1^{(s)}(q) \right) \circ \mathcal{F}_\nu(z)(i_1, \dots, i_{\nu-1}), \end{aligned}$$

where $\mathcal{N}_1^{(s)}(q)$ is 1-D NPT acting along the s -th coordinate direction and \otimes is the tensor product. The geometrical nature of spectrum $\hat{\mathcal{F}}_\nu(z)(k_1, \dots, k_{\nu-1})$ is obvious after substituting (3) into (4):

$$\begin{aligned} \hat{\mathcal{F}}_\nu(z)(k_1, \dots, k_{\nu-1}) &= \\ &= \sum_{i_1=0}^{q-1} \dots \sum_{i_{\nu-1}=0}^{q-1} f_\nu(i_1, \dots, i_{\nu-1}, i_\nu) z^{k_1 i_1 + \dots + k_{\nu-1} i_{\nu-1} + i_\nu} \\ &= \sum_{p=0}^{q-1} \left(\sum_{k_1 i_1 + \dots + k_{\nu-1} i_{\nu-1} + k_\nu = p} f_\nu(i_1, \dots, i_\nu) \right) z^p = \\ &= \sum_{p=0}^{q-1} \hat{f}_\nu(p, \{\alpha^\circ\}^\nu) z^p \pmod{(z^q - 1)}, \end{aligned} \quad (5)$$

where

$$\begin{aligned} \hat{f}_\nu(p, \{\alpha^\circ\}^\nu) &= \sum_{\{\{\alpha^\circ\}^\nu | \mathbf{i}\} = p} f_\nu(i_1, \dots, i_{\nu-1}, i_\nu) = \\ &= \sum_{k_1 i_1 + \dots + k_{\nu-1} i_{\nu-1} + k_\nu = p} f_\nu(i_1, \dots, i_{\nu-1}, i_\nu) \end{aligned} \quad (6)$$

and $\langle \{\alpha^\circ\}^\nu | \mathbf{i} \rangle = \{k_1 i_1 + \dots + k_{\nu-1} i_{\nu-1} + k_\nu = p\}$.

The coefficients of $\hat{\mathcal{F}}_\nu(z)(k_1, \dots, k_{\nu-1})$ are the spectrum of the Radon transform $\hat{f}_\nu(p, \{\alpha^\circ\}^\nu)$ of the initial signal $f_\nu(i_1, i_2, \dots, i_\nu)$ which are calculated using the fast NPT.

4. From the polynomial-valued spectrum we can obtain the classical Fourier spectrum:

$$F(ak_1, ak_2, \dots, ak_{\nu-1}, a) =$$

$$= \sum_{p=0}^{q-1} \widehat{f}_\nu(p, (k_1, k_2, \dots, k_{\nu-1}, 1)) W^{ap}. \quad (7)$$

Therefore, if $a_\nu \neq 0$ we can obtain Fourier spectrum of ν -D DFT lying on the rays $a\alpha^a = a \cdot (k_1, k_2, \dots, \dots, k_{\nu-1}, 1)$ from $\widehat{f}_\nu(p, \alpha_\nu^a)$ using $q^{\nu-1}$ of 1-D FFT.

Step 2.

1. Now we must calculate $\widehat{f}_\nu(p, \{\alpha^o\}^{\nu-1}) =$

$$= \sum_{k_1 i_1 + \dots + k_{\nu-2} i_{\nu-2} + k_{\nu-1} = p} \dots \sum_{i_\nu} f_\nu(i_1, i_2, \dots, i_\nu),$$

where $\{\alpha^o\}^{\nu-1} := (k_1, k_2, \dots, k_{\nu-2}, 1, 0)$. It is clear that $\widehat{f}_\nu(p, \{\alpha^o\}^{\nu-1}) =$

$$= \sum_{k_1 i_1 + \dots + k_{\nu-2} i_{\nu-2} + k_{\nu-1} = p} \dots \sum_{i_\nu} \left(\sum_{i_\nu=0}^{q-1} f_\nu(i_1, \dots, i_\nu) \right) =$$

$$= \sum_{k_1 i_1 + \dots + k_{\nu-2} i_{\nu-2} + k_{\nu-1} = p} \dots \sum_{i_{\nu-1}} f_{\nu-1}(i_1, i_2, \dots, i_{\nu-1}), \quad (8)$$

where $f_{\nu-1}(i_1, \dots, i_{\nu-1}) := \sum_{i_\nu=0}^{q-1} f_\nu(i_1, \dots, i_\nu)$.

2. Also, we have $(\nu-1)$ -D discrete Radon transform (8) of signal $f_{\nu-1}(i_1, \dots, i_{\nu-1})$. We repeat all the transformations with the signal $f_{\nu-1}(i_1, \dots, i_{\nu-1})$: that we have applied to the initial signal $f_\nu(i_1, \dots, i_{\nu-1}, i_\nu)$. Again we interpretate the $(\nu-1)$ -D scalar-valued signal $f_{\nu-1}(i_1, \dots, i_{\nu-1})$ as $(\nu-2)$ -D polynomial-valued signal:

$$\mathcal{F}_{\nu-1}(z)(i_1, \dots, i_{\nu-2}) = \sum_{i_{\nu-1}=0}^{q-1} f_{\nu-1}(i_1, \dots, i_{\nu-1}) z^{i_{\nu-1}}.$$

3. Let us find the polynomial-valued spectrum

$$\widehat{\mathcal{F}}_{\nu-1}(z)(k_1, \dots, k_{\nu-2}) =$$

$$= \sum_{i_1=0}^{q-1} \dots \sum_{i_{\nu-1}=0}^{q-1} \mathcal{F}_{\nu-1}(z)(i_1, \dots, i_{\nu-2}) z^{k_1 i_1 + \dots + k_{\nu-2} i_{\nu-2}} =$$

$$= \left(\bigotimes_{s=1}^{\nu-2} \mathcal{N}_1^{(s)}(q) \right) \circ \mathcal{F}_\nu(z)(i_1, \dots, i_{\nu-2}),$$

Obviously, $\widehat{\mathcal{F}}_{\nu-1}(z)(k_1, \dots, k_{\nu-2}) =$

$$= \sum_{p=0}^{q-1} \widehat{f}_{\nu-1}(p, \{\alpha^o\}^{\nu-1}) z^p \pmod{(z^q - 1)},$$

where $\widehat{f}_{\nu-1}(p, \alpha_{\nu-1}^o) =$

$$= \sum_{k_1 i_1 + \dots + k_{\nu-2} i_{\nu-2} + k_{\nu-1} = p} \dots \sum_{i_{\nu-1}} f_{\nu-1}(i_1, \dots, i_{\nu-2}, i_{\nu-1}).$$

4. From this polynomial-valued spectrum we can obtain Fourier transform lying on the rays $\{\alpha^o\}^{\nu-1} = \{(ak_1, ak_2, \dots, a, 0)\}$:

$$F((ak_1, \dots, a, 0)) = \sum_{p=0}^{q-1} \widehat{f}_{\nu-1}(p, (k_1, \dots, k_{\nu-2}, 1, 0)) W^{ap}$$

using $q^{\nu-2}$ of 1-D DFT, etc.....

Step $(\nu-1)$.

1. Repeating this process $\nu-1$ times we go to calculation following sum

$$\widehat{f}_2(p, \{\alpha^o\}^2) = \sum_{k_1 i_1 + 1 = p} \sum_{i_2} f_2(i_1, \dots, i_\nu).$$

It is clear that $\widehat{f}_2(p, \{\alpha^o\}^2) =$

$$= \sum_{k_1 i_1 + 1 = p} \left(\sum_{i_\nu=0}^{q-1} \dots \sum_{i_3=0}^{q-1} f_\nu(i_1, \dots, i_\nu) \right) = \sum_{k_1 i_1 + 1 = p} \sum_{i_2} f_2(i_1, i_2),$$

where

$$f_2(i_1, i_2) := \sum_{i_3=0}^{q-1} f_3(i_1, i_2, i_3).$$

2. Here we have 2-D discrete Radon transform of the signal $f_2(i_1, i_2)$. We interpret again this signal as 1-D polynomial-valued signal:

$$\mathcal{F}_2(z)(i_1) = \sum_{i_2=0}^{q-1} f_2(i_1, i_2) z^{i_2}.$$

3. The latter signal have polynomial-valued spectrum $\widehat{\mathcal{F}}_2(z)(k_1) :=$

$$= \sum_{i_1=0}^{q-1} \mathcal{F}_2(z)(i_1) z^{k_1 i_1} = \mathcal{N}_1(q) \{ \mathcal{F}_2(z)(i_1) \}.$$

4. Fourier spectrum of ν -D DFT lying on the rays $\{a(k_1, 1, 0, \dots, 0)\}$ has the form

$$F(ak_1, a, 0, \dots, 0) =$$

$$= \sum_{p=0}^{q-1} \left(\sum_{k_1 i_1 + k_2 = p} \sum_{i_2} f_2(i_1, i_2) \right) W^{ap} =$$

$$= \sum_{p=0}^{q-1} \widehat{f}_2(p, (k_1, 1, 0, \dots, 0)) W^{ap}.$$

We calculate using q of 1-D DFT.

Step ν .

And, finally, on the ν -th step we have to calculate one 1-D DFT

$$F(a \cdot (1, 0, 0, \dots, 0)) = \sum_{p=0}^{q-1} f_1(p) W^{pa},$$

where $f_1(i_1) = \sum_{i_2=0}^{q-1} f_2(i_1, i_2)$.

Now, let us summarize. We propose the following algorithm for computing DRT consisting of $\nu - 1$ steps of fast NPT:

Step 1. $\widehat{\mathcal{F}}_\nu(z)(k_1, \dots, k_{\nu-1}) =$
 $= \left(\bigotimes_{s=1}^{\nu-1} \mathcal{N}_1^{(s)}(q) \right) \circ \mathcal{F}_\nu(z)(i_1, \dots, i_{\nu-1}),$

Step 2. $\widehat{\mathcal{F}}_{\nu-1}(z)(k_1, \dots, k_{\nu-2}) =$
 $= \left(\bigotimes_{s=1}^{\nu-2} \mathcal{N}_1^{(s)}(q) \right) \circ \mathcal{F}_{\nu-1}(z)(i_1, \dots, i_{\nu-2}),$

.....,

Step $\nu - 2$. $\widehat{\mathcal{F}}_3(z)(k_1, k_2) =$
 $= \mathcal{N}_1^{(1)}(q) \otimes \mathcal{N}_1^{(2)}(q) \{ \mathcal{F}_3(z)(i_1, i_2) \},$

Step $\nu - 1$. $\widehat{\mathcal{F}}_2(z)(k_1) = \mathcal{N}_1^{(1)}(q) \{ \mathcal{F}_2(z)(i_1) \}$, where $\mathcal{N}_1^{(i)}(q)$ is 1-D q -points NPT acting along the i th coordinate direction.

The total number fast NPT equals

$$\sum_{i=\nu-1}^1 iq^{i-1} = \frac{q^{\nu-1}[\nu(q-1) - q] + 1}{(q-1)^2}$$

and if $q > 2$ this number is approximately $\nu q^{\nu-2}$.

For computing ν -D DFT $(q^{\nu-1} - 1)/(q - 1)$ 1-D fast DFT are required.

Recall [5] that

$$\mathbf{Ad}(\mathcal{N}_1(q)) = q\mathbf{Ad}(\mathcal{F}_1(q)), \mathbf{Mu}(\mathcal{N}_1(q)) = 0;$$

where

$$\mathbf{Ad}(\mathcal{N}_1(q)), \mathbf{Mu}(\mathcal{N}_1(q)) \text{ and } \mathbf{Ad}(\mathcal{F}_1(q)), \mathbf{Mu}(\mathcal{F}_1(q))$$

are additive and multiplicative complexities 1-D q -points fast NPT $\mathcal{N}_1(q)$ and DFT $\mathcal{F}_1(q)$, respectively.

In summary, the total complexity of the proposed algorithm for computing ν -D DRT is

$$\mathbf{Ad}(\mathcal{R}_\nu(q)) = \nu q^{\nu-2} \mathbf{Ad}(\mathcal{N}_1(q)), \mathbf{Mu}(\mathcal{R}_\nu(q)) = 0$$

and for computing ν -D DFT is

$$\mathbf{Ad}(\mathcal{F}_\nu(q)) = \nu q^{\nu-2} \mathbf{Ad}(\mathcal{N}_1(q)) +$$

$$+ \frac{q^{\nu-1} - 1}{q - 1} \mathbf{Ad}(\mathcal{F}_1(q)) \approx (\nu + 1)q^{\nu-1} \mathbf{Ad}(\mathcal{F}_1(q)),$$

$$\mathbf{Mu}(\mathcal{F}_\nu(q)) = \frac{q^{\nu-1} - 1}{q - 1} \mathbf{Mu}(\mathcal{F}_1(q)).$$

Therefore additive computer complexity of the present algorithm and traditional algorithms is equivalent, but multiplicative computer complexity of the new algorithm is ν times smaller.

4. CONCLUSIONS

The important contribution of this work is that it brings a new approach to independent/parallel decomposition of n -D Discrete Fourier Transform. This approach

- requires fewer number of 1-D DFT than the classical separable radix FFT-type approach,
- has ν times smaller multiplicative complexity of ν -D DFT compared to row/column approach,
- introduce a new fast direct and exact inversion algorithms of Discrete Radon Transform.

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