2-D HIGH RESOLUTION SPECTRAL ESTIMATION BASED ON MULTIPLE REGIONS OF SUPPORT

Stéphanie Rouquette[1] , Olivier Alata[2] , Mohamed Najim[1] and Charles W. Therrien[3]

[1] Equipe Signal et Image, ENSERB, B.P. 99, 33402 Talence Cedex, France
[2] IRCOM-SIC, UMR 6615, SP2MI, Téléport 2, B.P. 179, 86960 Futuroscope Cedex, France
[3] Dept. of Electrical and Computer Engr., Naval Postgraduate School, Monterey, CA, USA

ABSTRACT

This paper deals with frequency estimation in the 2-D case when one has only few data points. We propose a method to estimate the frequencies of a sum of exponentials. This method is based on an original set of 2-D linear prediction models with new regions of support derived from the standard quarter plane support region. These models define various spectra which are finally combined by computing their harmonic mean. This method benefits from the subspace decomposition of the covariance matrix to perform well. It is demonstrated that the new regions of support improve the spectrum geometry and the estimation accuracy compared to the classical quarter plane (QP) support regions.

1. INTRODUCTION

Linear prediction is a powerful and interesting tool in spectral analysis. In the 1-D case, Kumaresan and Tufts [3] proposed to estimate the signal frequencies from a subspace decomposition of the covariance matrix which leads to good results in comparison with the nonparametric methods. For 2-D signals, various linear prediction models have been developed. Four classical regions of support (quarter plane, nonsymmetric half plane, symmetric half plane and full plane) are referenced in litterature [4].
Recently new regions of support based on a modification of the standard quarter plane have been proposed and combined in [1] by Alata. These multiple models developed for a multichannel approach improve the estimation accuracy and the shape of the spectrum.
In this paper we propose a modification of this method to locate the frequencies of a 2-D signal modeled by a sum of complex exponentials in noise. We first present the signal model and the corresponding covariance matrix. We write the normal equations related to the new regions of support defined from the classical quarter plane. The harmonic mean of the corresponding spectra provides a new spectrum. We then develop a method to solve the normal equations in order to compute the spectral estimate. This method is based on the subspace decomposition of the covariance matrix and presents a low computational cost. We finally provide some simulation results to illustrate the improvement brought by the multiple support regions-based approach as compared to the classical QP one.

2. LINEAR PREDICTION

We begin with the following signal model:

$$\begin{cases} y(m,n) = x(m,n) + b(m,n), \\ 0 \leq m \leq M-1, \\ 0 \leq n \leq N-1, \end{cases} \tag{1}$$

where the noiseless signal x is defined by

$$x(m,n) = \sum_{k=1}^{K} a_k \exp[j2\pi(f_{1k}m + f_{2k}n) + j\varphi_k] \tag{2}$$

and the noise b is white gaussian with variance σ^2.

The data covariance matrix $\mathbf{R}$ is defined as follows:

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{(0)} & \mathbf{R}_{(-1)} & \cdots & \mathbf{R}_{(-P+1)} \\ \mathbf{R}_{(1)} & \mathbf{R}_{(0)} & \cdots & \mathbf{R}_{(-P+2)} \\ \vdots & \vdots & & \vdots \\ \mathbf{R}_{(P-1)} & \mathbf{R}_{(P-2)} & \cdots & \mathbf{R}_{(0)} \end{bmatrix}, \tag{3}$$

where

$$\mathbf{R}_{(m)} = \begin{bmatrix} r(m,0) & \cdots & r(m,-Q+1) \\ \vdots & & \vdots \\ r(m,Q-1) & \cdots & r(m,0) \end{bmatrix}. \tag{4}$$

The matrix $\mathbf{R}$ is Toeplitz - block Toeplitz with elements

$$\begin{aligned} r(p,q) &= r^*(-p,-q) \\ &= E\{y(m,n)y^*(m-p,n-q)\} \\ &= \sum_{k=1}^{K} a_k^2 e^{[j2\pi(f_{1k}p + f_{2k}q)]} + \sigma^2\delta(p,q) \end{aligned} \tag{5}$$

Consider now the 2-D linear prediction problem for the data (1). This takes the form

$$\hat{y}(m,n) = -\sum_{(p,q)} \sum_{\in\Omega} \nu_{p,q} y(m-p, n-q) \tag{6}$$

where $\hat{y}$ represents the predicted value of the signal from points within some region of support Ω. A typical region of support is the quarter plane shown in Fig. 1. For this support region, the coefficients are the solution to the normal equations:

$$\mathbf{R}\,\underline{\nu}_1 = \underline{e}_1. \tag{7}$$

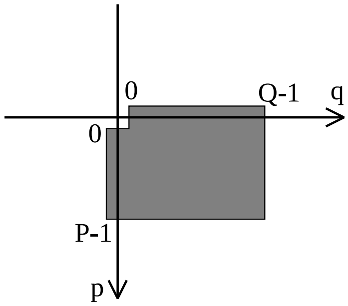

Figure 1: First Quarter Plane

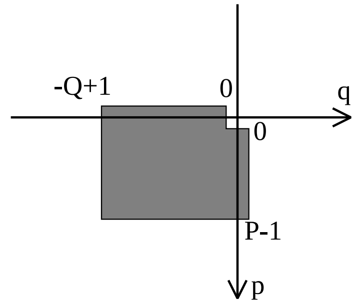

Figure 2: Second Quarter Plane

The coefficient vector $\underline{\nu}_1$ is given by the following relation:

$$\underline{\nu}_1 = \left[\nu_{0,0}^{(1)}, \cdots, \nu_{0,Q-1}^{(1)}, \cdots, \nu_{P-1,0}^{(1)}, \cdots, \nu_{P-1,Q-1}^{(1)}\right]^T \quad (8)$$

where $\underline{\nu}_1(1) = \nu_{0,0}^{(1)} = 1$, and the vector $\underline{e}_1$ is defined by

$$\underline{e}_1 = \left[\sigma_e^2, 0, \cdots, 0\right]^T \quad (9)$$

with σ_e^2 denoting the variance of the prediction error. The linear prediction filter can be used to locate complex exponentials by forming the following spectrum:

$$P_1(f_1, f_2) = \frac{\sigma_e^2}{\underline{s}^H(f_1, f_2)\underline{\nu}_1\underline{\nu}_1^H\underline{s}(f_1, f_2)} \quad (10)$$

where

$$\underline{s} = \left[1, \cdots, e^{j2\pi f_2(Q-1)}, \cdots, e^{j2\pi f_1(P-1)}, \cdots, e^{j2\pi(f_1(P-1)+f_2(Q-1))}\right]^T. \quad (11)$$

If a single quadrant filter is used, the spectral estimate has poor resolution and an asymmetric form. Therefore various researchers have been motivated to consider combined spectral estimates based on multiple regions of support [2] [5]. The simplest combination was proposed by Jackson and Chien [2] and is defined by

$$\frac{1}{P_{1,2}(f_1, f_2)} = \frac{1}{2}\left(\frac{1}{P_1(f_1, f_2)} + \frac{1}{P_2(f_1, f_2)}\right). \quad (12)$$

where the spectrum P_2 is related to the region of support shown in Fig. 2. Only two regions are necessary to compute the combined spectrum since the 1^{st} and 2^{nd} support regions produce identical results to the 3^{rd} and 4^{th} quadrants respectively.

To improve this method, Alata [1] has proposed the HVHM method where multiple support regions are taken in the framework of 2-D multichannel algorithms. In this paper we propose a modification of this method for subspace decomposition-based frequency estimation. This method leads to a very efficient computational procedure.
We consider a set of regions of support $\Omega_{H,l},\ l = 0, 1, \ldots, Q-1$, as represented in Fig. 3. For these support regions, the normal equations of linear prediction take the following form:

$$\mathbf{R}\,\underline{\nu}_{Hl} = \underline{e}_{Hl}, \quad (13)$$

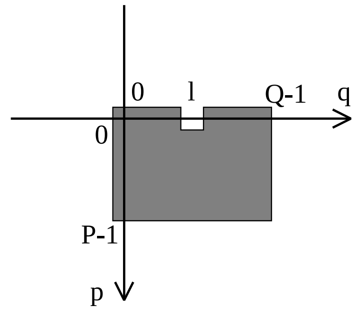

Figure 3: Region of support $\Omega_{H,l}$.

where the vector $\underline{e}_{Hl}$ and $\underline{\nu}_{Hl}$ are defined by

$$\underline{\nu}_{Hl} = \left[\nu_{0,0}^{(Hl)}, \cdots, \nu_{0,l}^{(Hl)}, \cdots, \nu_{P-1,0}^{(Hl)}, \cdots, \nu_{P-1,Q-1}^{(Hl)}\right]^T \quad (14)$$

$$\text{and } \underline{e}_{Hl} = \left[0, \cdots, \sigma_e^2, \cdots, 0\right]^T. \quad (15)$$

For the region $\Omega_{H,l}$, we have $\underline{\nu}_{Hl}(l+1) = \nu_{0,l}^{(Hl)} = 1$ and the term σ_e^2 appears in the corresponding location of $\underline{e}_{Hl}$. The spectral estimate related to this region is given by

$$P_{H,l}(f_1, f_2)\frac{\sigma_e^2}{\underline{s}^H(f_1, f_2)\underline{\nu}_{Hl}\underline{\nu}_{Hl}^H\underline{s}(f_1, f_2)}. \quad (16)$$

Likewise, we can define a region of support $\Omega_{V,l}$ as shown in Fig. 4.

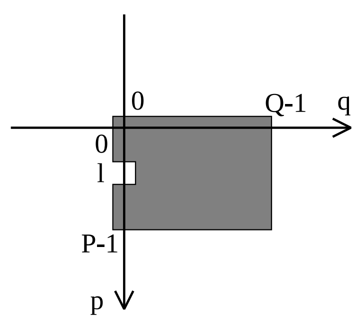

Figure 4: Region of support $\Omega_{V,l}$.

We consider the corresponding linear prediction problem and spectral estimate in a same way as previously. In that case we have

$$\underline{\nu}_{Vl}(lQ+1) = \nu_{l,0}^{(Vl)} = 1. \tag{17}$$

Finally, the combined spectral estimate computed from the support regions $\Omega_{H,l}$ and $\Omega_{V,l}$ is given by the following realtion:

$$\frac{1}{P_{H,V}(f_1,f_2)} = \frac{1}{P+Q}\times \left(\sum_{l=0}^{Q-1}\frac{1}{P_{H,l}(f_1,f_2)} + \sum_{l=0}^{P-1}\frac{1}{P_{V,l}(f_1,f_2)}\right). \tag{18}$$

3. SUBSPACE DECOMPOSITION BASED-SOLUTION OF NORMAL EQUATIONS

For the signal model (1), it is well known that the correlation matrix has the form

$$\mathbf{R} = \mathbf{S}\mathbf{\Psi}\mathbf{S}^H + \sigma^2\mathbf{I}, \tag{19}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{\Psi} = \mathrm{diag}\{a_k^2\}$. The k^{th} column of $\mathbf{S}$ is $\underline{s}(f_{1k}, f_{2k})$ and called $\underline{s}_k$.
The first and second terms of the decomposition of $\mathbf{R}$, $\mathbf{S}\mathbf{\Psi}\mathbf{S}^H$ and $\sigma^2\mathbf{I}$, respectively represent the noiseless signal and the noise covariance matrices. If $K \leq P$ and $K \leq Q$, the first one has rank K. The signal subspace is spanned by the columns $\underline{s}_k$ of $\mathbf{S}$, as well as by the eigenvectors $\underline{u}_k$ corresponding to the K largest eigenvalues d_k of $\mathbf{R}$. The noise subspace is spanned by the remaining eigenvectors which are orthogonal to those of the signal subspace. The data covariance matrix can therefore be written as

$$\mathbf{R} = \mathbf{U}_S\mathbf{D}_S\mathbf{U}_S^H + \sigma^2\mathbf{U}_N\mathbf{U}_N^H, \tag{20}$$

and its inverse is given by the following relation:

$$\mathbf{R}^{-1} = \mathbf{U}_S\mathbf{D}_S^{-1}\mathbf{U}_S^H + \frac{1}{\sigma^2}\mathbf{U}_N\mathbf{U}_N^H. \tag{21}$$

Let us consider first the regions of support $\Omega_{H,l}$. The corresponding normal equations (13) are equivalent to

$$\underline{\nu}_{Hl} = \left(\mathbf{U}_S\mathbf{D}_S^{-1}\mathbf{U}_S^H + \frac{1}{\sigma^2}\mathbf{U}_N\mathbf{U}_N^H\right)\underline{e}_{Hl}. \tag{22}$$

By left multiplying (22) by $\mathbf{U}_N\mathbf{U}_N^H$, we obtain

$$\mathbf{U}_N\mathbf{U}_N^H\underline{\nu}_{Hl} = \mathbf{U}_N\mathbf{U}_N^H\frac{1}{\sigma^2}\mathbf{U}_N\mathbf{U}_N^H\underline{e}_{Hl} \tag{23}$$

since the signal vectors are orthogonal to the noise vectors. Therefore we have

$$\underline{\nu}_{Hl} = \frac{\sigma_e^2}{\sigma^2}\mathbf{U}_N\underline{\tau}_{l+1} \tag{24}$$

where $\underline{\tau}_{l+1}$ denotes the $(l+1)^{\mathrm{th}}$ column of $\mathbf{U}_N^H$.
The relation $\underline{\nu}_{Hl}(l+1) = 1$ finally implies that the vector $\underline{\nu}_{Hl}$ is given by

$$\underline{\nu}_{Hl} = \frac{\underline{\nu}_{Hl}}{\underline{\nu}_{Hl}(l+1)} = \frac{\mathbf{U}_N\underline{\tau}_{l+1}}{\|\underline{\tau}_{l+1}\|^2} \tag{25}$$

with $\|\underline{\tau}_{l+1}\|^2 = \underline{\tau}_{l+1}^H\underline{\tau}_{l+1}$. The solution for the coefficient vector $\underline{\nu}_{Hl}$ is a scaled projection of $\underline{e}_{Hl}$ onto the noise subspace. Since $\underline{\nu}_{Hl}$ is orthogonal to all signal vectors, $P_{H,l}$ gives a clear peak at the signal values.

In addition, the knowledge of the prediction error variance is required to compute $P_{H,l}$. For the region of support $\Omega_{H,l}$, we have

$$\sigma_e^2 = \underline{\nu}_{Hl}^H\mathbf{R}\underline{\nu}_{Hl} \tag{26}$$

from (13). Combined with (25) this relation becomes

$$\sigma_e^2 = \frac{\underline{\tau}_{l+1}^H\mathbf{U}_N^H\mathbf{R}\,\mathbf{U}_N\underline{\tau}_{l+1}}{\|\underline{\tau}_{l+1}\|^4}, \tag{27}$$

which can be written as

$$\sigma_e^2 = \frac{\sigma^2}{\|\underline{\tau}_{l+1}\|^2} \tag{28}$$

if the eigenvectors are orthonormal. This equation shows the relation between the prediction error variance and the noise one. This latter can easily be estimated by the mean of the eigenvalues of $\mathbf{R}$ corresponding to the noise subspace.

The solution for the coefficient vector $\underline{\nu}_{Vl}$ is found in a similar way to be

$$\underline{\nu}_{Vl} = \frac{\mathbf{U}_N\underline{\tau}_{lQ+1}}{\|\underline{\tau}_{lQ+1}\|^2}, \tag{29}$$

and the variance of the prediction error corresponding to the support $\Omega_{V,l}$ is estimated as follows:

$$\sigma_e^2 = \frac{\sigma^2}{\|\underline{\tau}_{lQ+1}\|^2}. \tag{30}$$

4. SPECTRAL ESTIMATION RESULTS

In order to estimate the covariance matrix $\mathbf{R}$, we define a Hankel - block Hankel data matrix $\mathbf{Y}$ as

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_{(0)} & \mathbf{Y}_{(1)} & \cdots & \mathbf{Y}_{(M-P)} \\ \mathbf{Y}_{(1)} & \mathbf{Y}_{(2)} & \cdots & \mathbf{Y}_{(M-P+1)} \\ \vdots & \vdots & & \vdots \\ \mathbf{Y}_{(P-1)} & \mathbf{Y}_{(P)} & \cdots & \mathbf{Y}_{(M-1)} \end{bmatrix} \tag{31}$$

where $\mathbf{Y}_{(m)}$ is the following block:

$$\begin{bmatrix} y(m,0) & y(m,1) & \cdots & y(m,N-Q) \\ y(m,1) & y(m,2) & \cdots & y(m,N-Q+1) \\ \vdots & \vdots & & \vdots \\ y(m,Q-1) & y(m,Q) & \cdots & y(m,N-1) \end{bmatrix}.$$

The corresponding estimated covariance matrix is given by

$$\hat{\mathbf{R}} = \frac{1}{(M-P+1)(N-Q+1)}\mathbf{Y}\mathbf{Y}^H. \tag{32}$$

We first consider a signal with two components as described in Table 1. We estimated the spectra $P_{1,2}$ and $P_{H,V}$ which are respectively represented in Figs. 5 and 6. The multiple regions of support improve significantly the geometry of the spectrum which presents spurious peaks with a very small magnitude compared to those of the spectrum generated from the standard quarter plane models.
We also performed some Monte-Carlo simulations. The simulation conditions are given in Table 2. The Root Mean Square Error (RMSE) on the frequency f_{11} is plotted in Fig. 7 as a function of the Signal to Noise Ratio. The multiple regions of support improve the estimation accuracy for all values of SNR.

5. CONCLUSION

In this paper, we propose new models of 2-D linear prediction. The corresponding regions of support are based on a modification of the standard quarter plane. They provide a spectral estimate with better geometry than the classical QP support regions. In addition the frequency estimation accuracy is increased. To solve the normal equations, we take advantage of the data exponential model by using the subspace decomposition of the data covariance matrix .

6. ACKNOWLEDGEMENT

This work was supported in part by NATO under Grant CRG 970283.

7. REFERENCES

[1] O. Alata, P. Baylou and M. Najim , *A New 2-D Spectrum Estimate using Multichannel AR Approach of 2-D Fast RLS Algorithms*, Proc. of ICIP, pp. 442-445, October 1997.

[2] L. B. Jackson and H. C. Chien, *Frequency and Bearing Estimation by Two-Dimensional Linear Prediction*, Proc. of ICASSP, pp. 665-668, April 1979.

[3] R. Kumaresan and D. W. Tufts, *Estimating the angles of arrival of multiple plane waves*, IEEE Trans. on Aerospace and Electronic Systems, vol. 15, no. 1, pp. 134-139, January 1983.

[4] S. L. Marple Jr., *Digital Spectral Analysis with Applications*, Prentice-Hall, Englewood Cliffs, New Jersey, 1987.

[5] C. W. Therrien and H. T. El-Shaer, *Multichannel 2-D AR Spectrum Estimation*, IEEE Trans. on Acoustics, Speech and Signal Processing, vol. 37, no. 11, pp. 1798-1800, November 1989.

Frequencies f_{1k}	0.2	0.3
Frequencies f_{2k}	0.2	0.3
Magnitudes a_k	1	1
Phases φ_k	0	0
(M,N)=(15,15)		
(P,Q)=(6,6)		
SNR = 10 dB		

Table 1: Signal parameters.

Frequencies f_{1k}	0.22	0.28
Frequencies f_{2k}	0.24	0.25
Magnitudes a_k	1	1
Phases φ_k	0	0
(M,N)=(12,12)		
(P,Q)=(4,4)		
SNR: 10 to 20 dB by step of 1 dB		
Number of Monte-Carlo runs: 400		

Table 2: Simulation conditions.

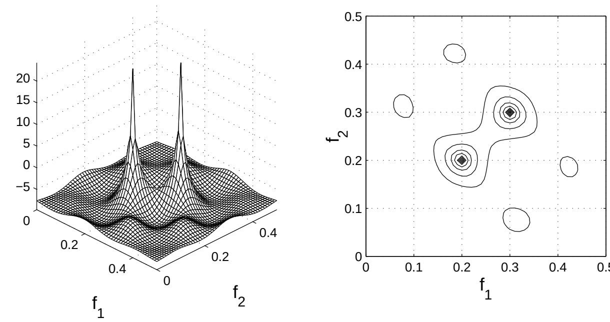

Figure 5: Spectrum $\hat{P}_{1,2}$ (dB).

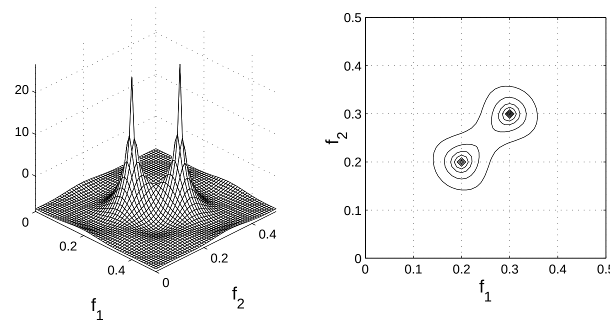

Figure 6: Spectrum $\hat{P}_{H,V}$ (dB).

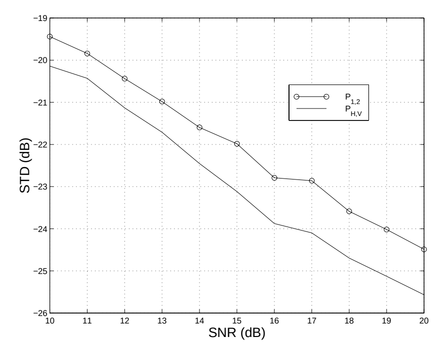

Figure 7: Estimation of f_{11}.