MODELING AND ESTIMATION OF MUTUAL COUPLING IN A UNIFORM LINEAR ARRAY OF DIPOLES

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ABSTRACT

The mutual coupling in a uniform linear array (ULA) of dipoles is calculated using basic electromagnetic concepts. Since the coupling often is unknown and needs to be estimated, a simpler model is proposed based on the electromagnetic analysis. The parameterization of this model is shown to be locally unambiguous. A necessary condition for the joint solution of directions and coupling parameters to be unique is also derived. Finally, the directions and coupling parameters are estimated using a maximum likelihood method. It is found that the simpler coupling model with just a few parameters well describes the full electromagnetic model.

1. INTRODUCTION

In the last two decades many high-resolution direction of arrival (DOA) estimation algorithms have been proposed with applications mainly in radar and sonar. Recently, the idea of using antenna arrays also in communication systems to increase the diversity has emerged. In practical antennas the elements of the array affect each other through mutual coupling and this reduces the direction finding ability. This is especially a problem with mass produced antennas in a rapidly changing environment such as base station antennas.

Much research has been performed on gain and phase uncertainties in the array response. However, on mutual coupling not much work has been presented. A modified MUSIC algorithm that includes a known coupling is presented in [8] and [5]. In [3], an iterative MUSIC method is presented that estimates an unknown coupling along with the DOAs, the gain and the phase.

Here, the mutual coupling in a uniform linear array of dipoles is calculated using basic electromagnetic concepts. This model for the coupling is used to examine how the presence of a known coupling affects the direction finding. However, this model contains too many unknown parameters when the coupling is unknown and a simpler model is therefore presented. The identifiability of that model is then investigated. Finally, the validity of the simpler model is examined by estimating both DOAs and coupling parameters using data generated by the full electromagnetic model.

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2. ARRAY MODEL AND COUPLING

The direction of an incident wave can be estimated by measuring the received voltages at the different elements of the array and by assuming the wave to be plane, simple geometric gives the direction. However, due to mutual coupling, the measured voltage at each element will depend not only on the incident field but also on the voltages on the other elements. The received voltage on each element will induce a current on the element which in turn radiates a field which affects the surrounding elements, i.e. mutual coupling. Here, mutual coupling in an array of n dipoles of finite length, l, is considered. To simplify the analysis the dipoles are considered thin, i.e. the radius $a << l$. The dipoles are placed linearly side-by-side with the same element separation, d , resulting in a ULA. The received voltages from this array, when p far-field narrow-band sources are used, are derived using basic electromagnetic concepts in [7]. The resulting model for the measured voltages is

$$\mathbf{x}(t) = \mathbf{C}\mathbf{A}(\boldsymbol{\phi})\mathbf{s}(t) + \mathbf{e}(t), \tag{1}$$

where the vector of measured voltages $\mathbf{x}(t)$ is $n \times 1$, the coupling matrix $\mathbf{C}$ is $n \times n$, the steering matrix $\mathbf{A}(\boldsymbol{\phi})$ is $n \times p$, the signal vector $\mathbf{s}(t)$ is $p \times 1$ and the noise vector $\mathbf{e}(t)$ is $n \times 1$. The DOAs are contained in the parameter vector $\boldsymbol{\phi}$. The steering matrix of a ULA has a Vandermonde structure with elements $[\mathbf{A}(\boldsymbol{\phi})]_{ij} = e^{-jkd(i-1)\cos\phi_j}$, where k is the wavenumber. The data model in (1) is identical to the usual data model used in array processing except for the coupling matrix

$$\mathbf{C} = (Z_A + Z_T)(\mathbf{Z} + Z_T\mathbf{I})^{-1}, \tag{2}$$

where Z_A is the antenna impedance, Z_T is the impedance of the measurement equipment at each element, and $\mathbf{Z}$ is the mutual impedance matrix. Deriving expressions for the mutual- and the antenna impedance is usually intractable but for thin and finite dipoles they can be calculated by using, for instance, the method of induced electro-motive force [1]. The impedance of the measurement equipment Z_T is chosen as the complex conjugate of the dipole impedance in order to reduce the powerloss. Other choices of Z_T is discussed in [6].

The model in (1) needs to be complemented with some additional assumptions which are used in the following sections.

- the coupling matrix has full rank, i.e., $\mathrm{rk}(\mathbf{C}) = n$ which implies that $\mathrm{rk}(\mathbf{CA}(\boldsymbol{\phi})) = \mathrm{rk}\,(\mathbf{A}(\boldsymbol{\phi})) = p$
- $\mathbf{e}(t)$ is circularly Gaussian distributed $E\{\mathbf{e}(t)\} = 0$, $E\{\mathbf{e}(t)\mathbf{e}^H(s)\} = \sigma^2\mathbf{I}\ \delta_{ts}$ and $E\{\mathbf{e}(t)\mathbf{e}^T(s)\} = \mathbf{0}\ \forall\ t, s$.
- $\mathbf{s}(t)$ is also circularly Gaussian distributed $E\{\mathbf{s}(t)\} = 0$, $E\{\mathbf{s}(t)\mathbf{s}^H(s)\} = \mathbf{S}\ \delta_{ts}$ and $E\{\mathbf{s}(t)\mathbf{s}^T(s)\} = \mathbf{0}\ \forall\ t, s$.

3. DOA ESTIMATION WITH KNOWN COUPLING

The problem of estimating the DOA in the presence of a known coupling is not very different from the usual DOA estimation problem. Most algorithms which has appeared in the last decades are possible to modify to be able to cope with mutual coupling. Basically it is just to change the steering matrix $\mathbf{A}$ to an effective steering matrix $\mathbf{CA}$. However, if the algorithm uses some special structure of the steering matrix it can not, at least not directly, be used when mutual coupling is present. In [5, 8] a modified version of the popular MUSIC algorithm has been proposed. This version will be briefly reviewed and also two other algorithms that straightforwardly can be extended to include a known coupling.

The fundamental idea of MUSIC is based on a subspace approach [4]. The eigenvectors of the covariance matrix of the measured voltages are divided in to signal eigenvectors, $\mathbf{E}_s$, which span the signal subspace and noise eigenvectors, $\mathbf{E}_n$, which span the noise subspace. Using the properties of the covariance matrix it can be shown that the steering vector is perpendicular to the noise subspace. Mutual coupling is included by simply inserting the effective steering vector $\mathbf{a}^H(\phi)\mathbf{C}^H\mathbf{E}_n = 0$. The MUSIC algorithm is usually formulated as searching for the p largest peaks of the spatial spectrum

$$P_{MU}(\phi) = \frac{1}{\mathbf{a}^H(\phi)\mathbf{C}^H\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^H\mathbf{C}\mathbf{a}(\phi)}, \tag{3}$$

where $\hat{\mathbf{E}}_n$ is obtained from the sample covariance matrix. Another closely related method for estimating DOAs is Root-Music, that straightforwardly can be modified to include coupling. This method differs from the above in that it requires the antenna to be a ULA. Define the steering vector $[\mathbf{a}(z)]_i = z^{i-1}$. Then, find the roots of the polynomial

$$f(z) = \mathbf{a}^T(z^{-1})\mathbf{C}^H\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^H\mathbf{C}\mathbf{a}(z). \tag{4}$$

Pick the p roots closest to the unit circle, $\hat{z}_i$ and calculate the angle and solve for ϕ_i in $\angle\hat{z}_i = \angle e^{-jkd\cos\phi_i}$.

A different method that is easily adopted to mutual coupling is ESPRIT [4], that can be used in arrays with a translational invariance structure. ESPRIT exploits this invariance and the fact that $\mathcal{R}\{\mathbf{E}_s\} = \mathcal{R}\{\mathbf{A}(\boldsymbol{\phi})\}$. With mutual coupling present, the relation simply becomes $\mathcal{R}\{\mathbf{C}^{-1}\mathbf{E}_s\} = \mathcal{R}\{\mathbf{A}(\boldsymbol{\phi})\}$. Here, the invariance structure is obtained by using maximum overlapping sub-arrays.

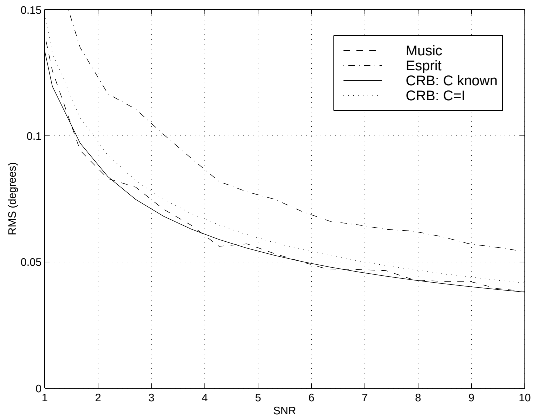

Figure 1: CRB and RMS error of Root-Music and ESPRIT for ϕ_1 using an array of 10 $\lambda/2$ dipoles spaced $\lambda/2$ apart when two waves are incident from $(80°, 85°)$ and 1000 snapshots with SNR $= 20dB$.

The estimation performance of the above methods is examined by Monte-Carlo simulations using 500 trials. In Figure 1, the RMS errors of the modified MUSIC and ESPRIT methods are shown. The Cramér-Rao lower bound (CRB) is also shown for the case with coupling and without. The MUSIC method is close to

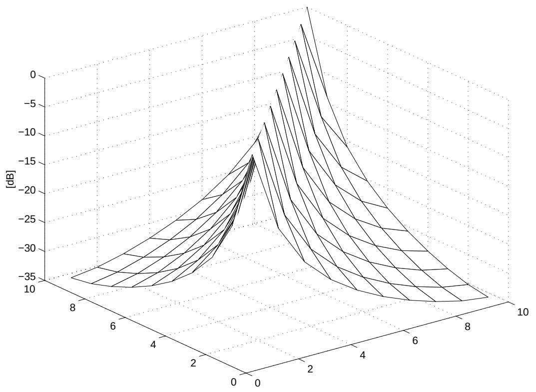

Figure 2: The magnitude of the elements of the coupling matrix of an array of 10 $\lambda/2$ dipoles spaced $\lambda/2$ apart.

the CRB, but ESPRIT performs worse, since it only exploits the invariance structure and not the full matrix. The performance is thus the same as in the coupling free case. However, it is interesting to observe that the CRB when a known coupling is present is slightly lower than the coupling free case. This is not generally true, but the difference is usually small [7].

4. PARAMETERIZATION AND IDENTIFIABILITY

The methods in the previous section required that the coupling matrix was known. Often this is not the case and the coupling needs to be estimated along with the DOAs. The model for the coupling matrix in (2) contains n^2 complex entries and a more efficient parameterization of the coupling is therefore desired. In Figure 2, the absolute value of the elements of the full electromagnetic $\mathbf{C}$ is shown. From this figure it is clear that the coupling between neighboring elements is almost the same along the array, thus the number of parameters to be estimated can be reduced by a factor of n. By normalizing the main diagonal to unity, the following coupling model results

$$\mathbf{C}_{ij} = c_{|i-j|} \quad \text{where} \quad c_0 = 1. \tag{5}$$

The magnitude of the coupling parameters decreases quite fast, so that it is often enough with just a few (q) nonzero c_k (or sub-diagonals).

Before discussing methods of estimating the unknown coupling along with the DOAs, the parameter identifiability of the problem needs to be examined. When the signal $\mathbf{s}$ is known, for example a training sequence, the problem of parameter identifiability reduces to show that the parameterization is unambiguous, i.e.

$$\mathbf{C}(\mathbf{c})\mathbf{A}(\boldsymbol{\phi}) = \mathbf{C}(\mathbf{c}')\mathbf{A}(\boldsymbol{\phi}') \Leftrightarrow \begin{matrix} \mathbf{c} = \mathbf{c}' \\ \boldsymbol{\phi} = \boldsymbol{\phi}' \end{matrix}. \tag{6}$$

This corresponds to showing that $\mathbf{C}(\mathbf{c})\mathbf{A}(\boldsymbol{\phi}) = y$ has a unique solution, and since there exists at least one solution the analytical implicit function theorem [2] can be used to show that this is the only solution. First, note that it is enough to consider one column of the steering matrix since if (6) holds for one column it will hold for all columns. Secondly, the array is a ULA and thus the steering vector can be written as $\mathbf{a}_i = \lambda^{i-1}$, where $\lambda = e^{-jkd\cos\phi}$. Then $\mathbf{C}(\mathbf{c})\mathbf{A}(\lambda) = y$ represents a system of equations in λ and c_i $i = 1 \ldots q$. If the Jacobian of this system of equations is non-zero, then the analytical implicit function theorem ensures the existence of a unique function in the neighborhood of the true parameter values and (6) holds locally. The Jacobian can be regarded as a polynomial in λ, and for it to be zero the polynomial has to have zeros at the unit circle. However, by examining this polynomial it is found that the zeroth order coefficient is c_1 and the highest order coefficient is $n \Leftrightarrow 1$. For a polynomial to have zeros on the unit circle the coefficient vector must be conjugate-symmetric and thus $c_1 = n \Leftrightarrow 1$. Since $|c_1| < 1$ and $n \geq 2$, the polynomial can not have zeros on the unit circle and thus (6) holds at least locally.

Usually the signals are not known, and in this case a necessary condition for the existence of a unique solution for $\boldsymbol{\phi}$ and $\mathbf{c}$ can be derived from the relation

$$\mathbf{E}_s = \mathbf{C}(\mathbf{c})\mathbf{A}(\boldsymbol{\phi})\mathbf{T}, \tag{7}$$

which is used in the so-called Subspace Fitting methods [4], and contains all information about the DOAs and the coupling parameters. The number of independent real equations are $2np'$, where p' is the rank of signal covariance $\mathbf{S}$. The number of real unknowns in (7) are $2q + p$ from $\mathbf{CA}$ and $2pp'$ from $\mathbf{T}$. If

$$n \geq p + \frac{p}{2p'} + \frac{q}{p'} \tag{8}$$

does not hold, there are more unknowns than equations. By the implicit function theorem [2], assuming that the conditions for its existence hold, an infinite number of solutions exists. Thus, (8) is a necessary condition for uniqueness.

5. JOINT ESTIMATION OF DOA AND COUPLING

Most multidimensional search methods, stochastic maximum likelihood (SML) etc., can easily be modified to

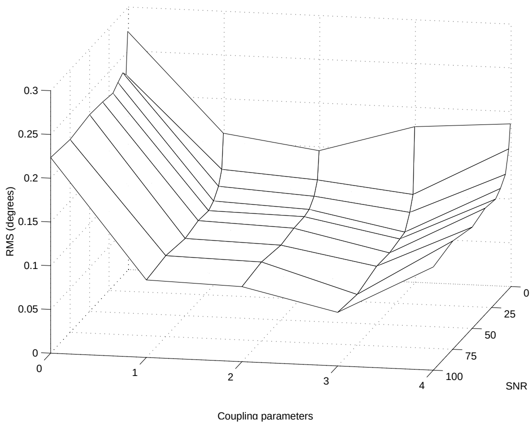

Figure 3: The RMS error for $\phi = 80°$ when jointly estimating DOA and coupling using the array in Figure 2 when two waves are incident from $(80°, 85°)$ and 1000 snapshots.

include coupling. However, when the coupling parameters are included, the initialization of the search becomes more important. A search can be avoided by using an iterative version of the MUSIC algorithm that estimates DOAs, mutual coupling, gain, and phase [3]. However, the convergence to the true parameter values is very slow and the method also suffers from the resolution threshold of MUSIC for closely spaced DOAs.

Here, the SML method is used instead since it required much less computation time. The initialization, however, was obtained by using an iterative Root-Music algorithm in the same fashion as the MUSIC method in [3].

Now, it is interesting to estimate the coupling and the DOAs with the simple model in (5) using data generated using the full electromagnetic model in (2). The RMS error when jointly estimating DOA and coupling, as two waves impinge on the array, is shown in Figure 3. Five iterations of the above mentioned Root-Music method was used to obtain initialization values for SML. Estimating a single coupling parameter more than halves the RMS error, which mainly is due to bias. When more parameters are estimated the level of the bias depends on the DOAs and the number of elements. As more and more coupling parameters are included the variance increases and there is thus a compromise between bias and variance. In this case three parameters is a reasonable choice. Also note that the RSM error decreases with an increase in the SNR until the bias dominates, which for example, is the case when no coupling parameters are estimated. Thus, by estimating a only few coupling parameters the RMS error can be made much smaller and the coupling matrix in (2) with 100 entries is in this case well modeled by only 3 parameters.

6. CONCLUSIONS

The mutual coupling was calculated for a ULA of dipoles using basic electromagnetic concepts. Since the coupling often is unknown and needs to be estimated, a simpler model was proposed based on the electromagnetic analysis. The parameterization of this model was shown to be locally unambiguous. Also, a necessary condition for the solution of the DOAs and the coupling parameters to be unique was derived.

Finally, the validity of the simpler coupling model was examined by jointly estimate both DOAs and coupling using data generated by the full electromagnetic model. It was found that the coupling, calculated using electromagnetics, was well described by the simpler model with just a few parameters.

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