SECOND MOMENT ANALYSIS OF THE FILTERED-X LMS ALGORITHM*

Orlando J. Tobias[1], José C.M. Bermudez[1] , Neil J. Bershad[2] and Rui Seara[1]

[1] Department of Electrical Engineering
Universidade Federal de Santa Catarina, Florianópolis, SC, BRAZIL

[2]Department of Electrical and Computer Engineering
University of California, Irvine, CA, USA
E-mails: orlando@linse.ufsc.br, bermudez@fastlane.com.br, bershad@ece.uci.edu, seara@linse.ufsc.br

ABSTRACT

This paper presents a new analytical model for the second moment behavior of the Filtered-X LMS algorithm. The new model is not based on the independence theory, and is derived for gaussian inputs and slow adaptation. Monte Carlo simulations show excellent agreement with the behavior predicted by the theoretical model.

1. INTRODUCTION

Active noise and vibration control (ANC) has become an important application area for adaptive filters. The most popular adaptive algorithm used for ANC is the Filtered-X LMS (FXLMS) algorithm [1,2]. This algorithm is a modification of the well known LMS algorithm. The reference signal is filtered so as to compensate for filters inherent in the electro-acoustic adaptation loop. These additional filters in the adaptation system significantly complicate the behavior and analysis of the adaptive algorithm. Conventional LMS analysis is not applicable to the FXLMS algorithm. Also, simplifying LMS assumptions cannot easily be extended to FXLMS. This applies to the well known independence theory (IT), in which time-lagged input vectors are assumed independent. The additional loop filters generate signal correlations. These correlations invalidate the IT for simplifying the statistical analysis of the FXLMS. Hence, the exact analysis of FXLMS becomes very complex. This complexity holds even for the exact analysis of the conventional LMS algorithm [3]. Most of the FXLMS stochastic analyses in the open literature concentrate upon algorithm stability [2-5]. Stability results are useful for algorithm design. However, transient and steady-state weight statistical behavior under different implementation conditions is necessary for a better understanding of the algorithm's properties.

Some results have been obtained recently for the stochastic behavior of FXLMS. In particular, a recursion was derived in [4] for the mean weight behavior. The analysis assumed both IT and extremely slow convergence rates. First and second moment adaptive weight expressions were derived using IT in [6] for the case of exact secondary path estimation. The expressions were then specialized to the Delayed LMS algorithm. These results yielded some insight into FXLMS behavior. However, the model and conclusions for the Delayed LMS algorithm cannot easily be extended to non-trivial secondary path filters. The signal correlations caused by the ANC system filters invalidate the IT for mathematical simplification. Hence, IT cannot be used and the exact analysis becomes mathematically intractable.

A more recent result [5] has presented an analytical model for the mean weight behavior of the FXLMS algorithm that does not use the independence theory. The simplifying assumption used in [5] was that the correlation between reference signal vectors is more important for determining the weight vector behavior than the correlations between the weight and reference signal vectors. This assumption is supported by extensive numerical simulations.

This paper presents a second moment analysis of the FXLMS algorithm for the case of exact secondary path estimation. The analysis does not rely on the IT. A deterministic recursion is obtained for the weight-error covariance matrix, under the same assumption used in [5] and for a gaussian reference input. This recursion is then used to predict the mean square error behavior. Monte Carlo simulations show excellent agreement with the theoretical predictions in both the transient phase of adaptation and in steady-state.

2. ANALYSIS

The Analysis Model

Fig. 1 shows a block diagram for the FXLMS algorithm. $\mathbf{W^o}$ is the unknown system, $\mathbf{W}(n)$ is the adaptive filter and $\hat{\mathbf{S}}$ and $\hat{\mathbf{S}}$ are linear filters. This diagram can model a ANC system [1] and $\mathbf{S}$ is often denoted the secondary path filter. Usually, $\hat{\mathbf{S}}$ is designed to duplicate $\mathbf{S}$, and this is the case analyzed here.

The notation used in this paper is the following:

$\mathbf{W^o} = \left[w_0^o, w_1^o, \cdots, w_{N-1}^o\right]^T$: impulse response to be identified

$\mathbf{W}(n) = \left[w_0(n), w_1(n), \cdots, w_{N-1}(n)\right]^T$: adaptive weight vector

$\mathbf{S} = \left[s_0 \;\; s_1 \;\; \cdots \;\; s_{M-1}\right]^T$: impulse response of the system in the auxiliary path

$\mathbf{X}(n) = \left[x(n), \cdots, x(n-N+1)\right]^T$: observed data vector

* This work was supported in part by CAPES (Brazilian Ministry of Education) and by CNPq (Brazilian Ministry of Science and Technology) under Grant No. 352084/92-8.

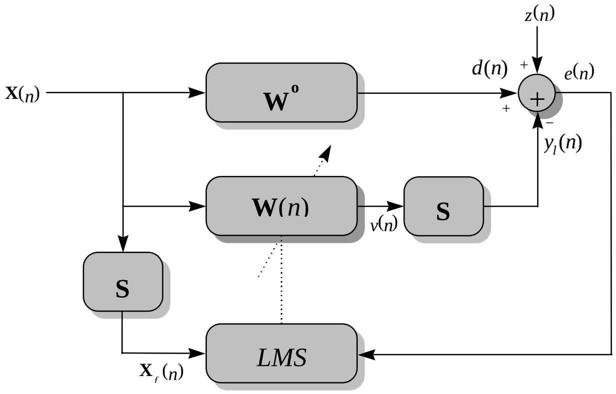

Figure 1. Block diagram - FXLMS algorithm.

$\mathbf{X}_f(n) = \left[x_f(n), \cdots, x_f(n-N+1)\right]^T$: filtered reference signal

$d(n)$: primary signal

$z(n)$: white noise (uncorrelated with any other signal) with variance σ_z^2.

For the analysis, $x(n)$ is assumed gaussian with variance σ_x^2. Also, the dimensions of $\mathbf{W}^{\mathbf{o}}$ and $\mathbf{W}(n)$ are assumed the same for notational simplicity.

3. MEAN-SQUARE ERROR EQUATION

For $\hat{\mathbf{S}} = \mathbf{S}$, the signals in Fig. 1 can be described by:

$$e(n) = d(n) - y_l(n) + z(n) \tag{1}$$

$$d(n) = \mathbf{X}^T(n)\mathbf{W}^{\mathbf{o}} \tag{2}$$

$$y(n) = \mathbf{X}^T(n)\mathbf{W}(n) = \mathbf{W}^T(n)\mathbf{X}(n) \tag{3}$$

$$y_l(n) = \sum_{i=0}^{M-1} s_i\, y(n-i) = \sum_{i=0}^{M-1} s_i\, \mathbf{X}^T(n-i)\mathbf{W}(n-i) \tag{4}$$

$$x_f(n) = \sum_{i=0}^{M-1} s_i\, x(n-i) \tag{5}$$

$$\mathbf{X}_f(n) = \sum_{i=0}^{M-1} s_i\, \mathbf{X}(n-i) \tag{6}$$

Substituting (2)-(4) in (1) yields the instantaneous error

$$e(n) = d(n) - \sum_{i=0}^{M-1} s_i\, \mathbf{X}^T(n-i)\mathbf{W}(n-i) + z(n) \tag{7}$$

Squaring (7) yields

$$\begin{aligned} e^2(n) &= d^2(n) - 2\sum_{i=0}^{M-1} s_i d(n)\mathbf{X}^T(n-i)\mathbf{W}(n-i) \\ &+ \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} s_i s_j \mathbf{X}^T(n-i)\mathbf{W}(n-i)\mathbf{X}^T(n-j)\mathbf{W}(n-j) \\ &+ 2d(n)z(n) - 2\sum_{i=0}^{M-1} s_i z(n)\mathbf{X}^T(n-i)\mathbf{W}(n-i) + z^2(n). \end{aligned} \tag{8}$$

Taking the expected value of (8) yields the expression for he mean square error (MSE):

$$\begin{aligned} E\left[e^2(n)\right] &= E\left[d^2(n)\right] - 2\sum_{i=0}^{M-1} s_i E\left[d(n)\mathbf{X}^T(n-i)\mathbf{W}(n-i)\right] \\ &+ \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} s_i s_j E\left[\mathbf{X}^T(n-i)\mathbf{W}(n-i)\mathbf{X}^T(n-j)\mathbf{W}(n-j)\right] \\ &- 2\sum_{i=0}^{M-1} s_i E\left[z(n)\mathbf{X}^T(n-i)\mathbf{W}(n-i)\right] \\ &+ 2E\left[d(n)z(n)\right] + E\left[z^2(n)\right]. \end{aligned} \tag{9}$$

It has been shown in [4], [5], [7] that the optimum weight vector which minimizes (9) is given by:

$$\mathbf{W}_{opt} = \tilde{\mathbf{R}}_{ss}^{-1}\tilde{\mathbf{P}}_s \tag{10}$$

where $\tilde{\mathbf{R}}_{ss} = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} s_i s_j \mathbf{R}_{j-i}$, $\mathbf{R}_{j-i} = E\left[\mathbf{X}(n-i)\mathbf{X}^T(n-j)\right]$, $\tilde{\mathbf{P}}_s = \sum_{i=0}^{M-1} s_i \mathbf{P}_i$ and $\mathbf{P}_i = E\left[d(n)\mathbf{X}(n-i)\right]$.

To proceed with the analysis, it is assumed that the correlation between reference signal vectors is more important than the correlation between weight and reference signal vectors. This assumption is supported by extensive numerical simulations.

Defining the weight-error vector $\mathbf{V}(n) = \mathbf{W}(n) - \mathbf{W}_{opt}$, using (2) an assuming weight and reference signal vectors independent, the MSE equation (9) becomes:

$$\begin{aligned} E\left[e^2(n)\right] &= \mathbf{W}^{o^T}\mathbf{R}_0\mathbf{W}^o - \mathbf{W}^{o^T}\left(\sum_{i=0}^{M-1} s_i \mathbf{R}_i\right)\mathbf{W}_{opt} \\ &+ 2\mathbf{W}_{opt}^T \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} s_i s_j \mathbf{R}_{j-i} E\left[\mathbf{V}(n-i)\right] \\ &- 2\mathbf{W}^{o^T}\sum_{i=0}^{M-1} s_i \mathbf{R}_i E\left[\mathbf{V}(n-i)\right] \\ &+ \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} s_i s_j tr\left\{\mathbf{R}_{j-i} E\left[\mathbf{V}(n-j)\mathbf{V}^T(n-i)\right]\right\} + \sigma_z^2. \end{aligned} \tag{11}$$

A recursive expression for $E\left[\mathbf{V}(n-i)\right]$ can be readily obtained from the results in [5] as

$$E\left[\mathbf{V}(n+1)\right] = E\left[\mathbf{V}(n)\right] - \mu\left\{\sum_{i=0}^{M-1}\sum_{j=0}^{M-1} s_i s_j \mathbf{R}_{i-j} E\left[\mathbf{V}(n-i)\right]\right\} \tag{12}$$

Thus, the only undetermined term in (11) is the weight-error covariance matrix $E[\mathbf{V}(n-j)\mathbf{V}^T(n-i)]$. A recursive equation to determine this matrix is derived in the next section.

4. WEIGHT-ERROR COVARIANCE MATRIX

A recursion for the adaptive weight vector can be obtained from the results in [5] as:

$$\mathbf{W}(n-j)=\mathbf{W}(n-j-1)+\mu\Bigg(\sum_{k=0}^{M-1}s_j\mathbf{X}(n-k-j-1)\mathbf{X}^T(n-j-1)\mathbf{W}^{\mathbf{o}} - \sum_{k=0}^{M-1}\sum_{l=0}^{M-1}s_ls_j\mathbf{X}(n-l-j-1)\mathbf{X}^T(n-k-j-1)\mathbf{W}(n-k-j-1) + \sum_{l=0}^{M-1}s_j\mathbf{X}(n-l-j-1)z(n-j-1)\Bigg). \tag{13}$$

Subtracting $\mathbf{W}_{opt}$ from both sides of (13) and using the definition of the weight-error vector $\mathbf{V}(n)$ yields:

$$\mathbf{V}(n-j)=\mathbf{V}(n-j-1)+\mu\Bigg(\sum_{k=0}^{M-1}s_k\mathbf{X}(n-k-j-1)\mathbf{X}^T(n-j-1)\mathbf{W}^{o} - \sum_{k=0}^{M-1}\sum_{l=0}^{M-1}s_ks_l\mathbf{X}(n-l-j-1)\mathbf{X}^T(n-k-j-1)\mathbf{W}_{opt} - \sum_{k=0}^{M-1}\sum_{l=0}^{M-1}s_ks_l\mathbf{X}(n-l-j-1)\mathbf{X}^T(n-k-j-1)\mathbf{V}(n-k-j-1) + \sum_{l=0}^{M-1}s_l\mathbf{X}(n-l-j-1)z(n-j-1)\Bigg). \tag{14}$$

Post-multiplying (14) by its transpose, taking the expected value, assuming weight and reference input vectors independent and using the fourth-order moment theorem for a gaussian process, the expression for the weight-error covariance matrix is obtained as

$$\begin{aligned}
E\left[\mathbf{V}(n-j)\mathbf{V}^T(n-i)\right] &= E\left[\mathbf{V}(n-j-1)\mathbf{V}^T(n-i-1)\right] \\
&- \mu\sum_{k=0}^{M-1}\sum_{l=0}^{M-1}s_ks_lE\left[\mathbf{V}(n-j-1)\mathbf{V}^T(n-k-i-1)\right]\mathbf{R}_{l-k} \\
&- \mu\sum_{k=0}^{M-1}\sum_{l=0}^{M-1}s_ks_l\mathbf{R}_{k-l}E\left[\mathbf{V}(n-k-j-1)\mathbf{V}^T(n-i-1)\right] \\
&+ \mu^2\sum_{k=0}^{M-1}\sum_{l=0}^{M-1}\sum_{m=0}^{M-1}\sum_{r=0}^{M-1}s_ks_ls_ms_r\Big(\mathbf{R}_{k-l}E\left[\mathbf{V}(n-k-j-1)\mathbf{V}^T(n-m-i-1)\right]\mathbf{R}_{r-m} \\
&\qquad + \mathbf{R}_{m+i-l-j}E\left[\mathbf{V}(n-m-i-1)\mathbf{V}^T(n-k-j-1)\right]\mathbf{R}_{r+i-k-j} \\
&\qquad + \mathbf{R}_{r+i-l-j}tr\{E\left[\mathbf{V}(n-m-i-1)\mathbf{V}^T(n-k-j-1)\right]\mathbf{R}_{m+i-k-j}\}\Big) \\
&+ \mu^2\sum_{k=0}^{M-1}\sum_{l=0}^{M-1}s_ks_l\mathbf{R}_{l+i-k-j}E\left[z(n-i-1)z(n-j-1)\right].
\end{aligned} \tag{15}$$

Recursions (11), (12) and (15) define the analytical model for the second moment behavior of the FXLMS algorithm.

5. SIMULATION RESULTS

This section presents some examples which verify the accuracy of the deterministic model composed by equations (11), (12) and (15). Consider the following cases:

a) $\mathbf{W}^o=[-0.0197,\ 0.1179,\ 0.8,\ 0.1179,\ -0.0197]$, $\mu=10^{-3}$, $\mathbf{S}=[1.0,\ 0.5]$, $x(n)$ white and gaussian, $\sigma_x^2=1$, $\sigma_z^2=0.001$.

b) $\mathbf{W}^o=[-0.0197,\ 0.1179,\ 0.8,\ 0.1179,\ -0.0197]$, $\mu=10^{-3}$, $\mathbf{S}=[1.0,\ -0.5]$, $x(n)$ white and gaussian, $\sigma_x^2=1$, $\sigma_z^2=0.001$.

These two examples were chosen to show the effect of secondary path filter frequency response on the algorithm behavior. The filter **S** is low-pass in (a) and a high-pass in (b).

Fig. 2 shows the MSE behavior predicted by the theoretical model and obtained from a Monte Carlo simulations (100) runs for example (a). The two curves are in excellent agreement. The curves show excellent agreement. Fig. 3 shows the evolution of selected elements of the present weight-error covariance matrix obtained from simulation and calculated using (15). Again, the theoretical predictions are in close agreement with the simulation results. The results were plotted at intervals of 75 iterations for better visualization. This suggests that the theoretical assumptions are accurate. Similar results were obtained for the other weight-error correlation matrices.

Curves for example (b) are shown in Figs. 4 and 5 (corresponding to Figs. 2 and 3). The results in Fig. 5 were also plotted at intervals of 75 iterations. Again, the theoretical and simulated results are in excellent agreement.

These examples clearly show that the new model can be used to study and predict properties of the FXLMS algorithm. The secondary path frequency response can have a significant effect on the cancellation level as shown by these two examples. As compared to LMS cancellation, the low-pass filter causes a 10 dB cancellation loss whereas the high-pass filter causes about a 3 dB loss.

The theoretical model considers the effects of the signal correlations introduced by the filtering operations. Thus, each update of (15) or (11) requires all weight-error covariance matrices since system initialization. This should come at no surprise and happens also for models of the LMS algorithm that include signal correlation effects [3]. Three past terms were used to compute the covariance matrices for the examples presented here.

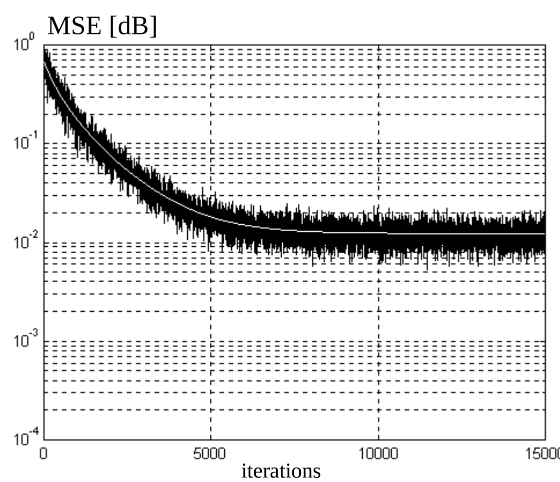

Fig. 2. MSE: simulation and theoretical prediction for example (a) .

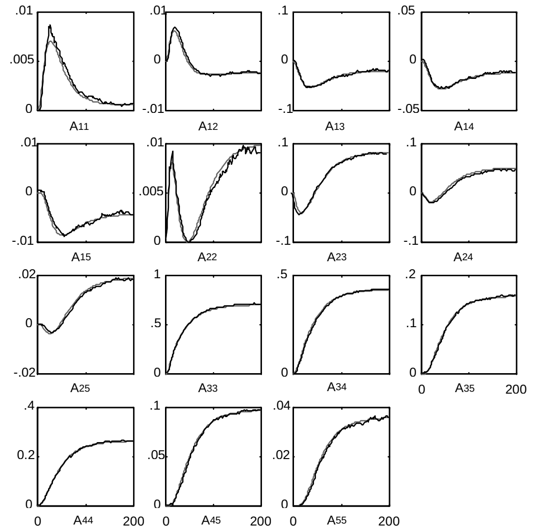

Fig. 3. Simulation and theoretical predictions of time evolution for selected elements $A_{ij} = E[\mathbf{V}(n-1)\mathbf{V}^T(n-1)]_{ij}$ from example (a)

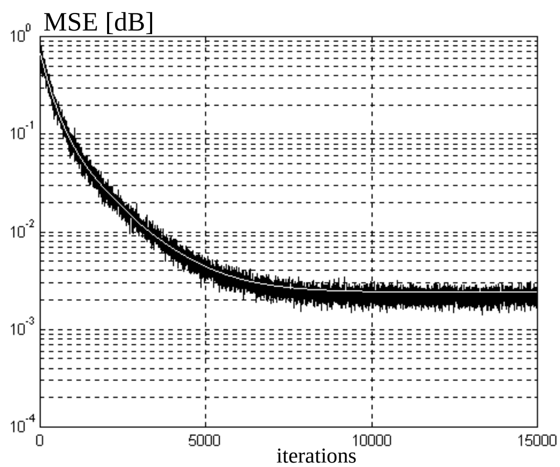

Fig. 4. MSE: simulation and theoretical prediction for example (b).

6. CONCLUSIONS

This paper has presented a second moment analysis of the FXLMS algorithm for gaussian inputs. The analysis included the correlations between lagged input vectors and was not based on independence theory. An analytical model has been derived to predict the algorithm behavior both during the transient phase of adaptation and in steady-state. This model consists of deterministic recursions for the MSE and weight-error covariance matrix. Monte Carlo simulations show excellent agreement with the theoretical predictions obtained from the new analytical model.

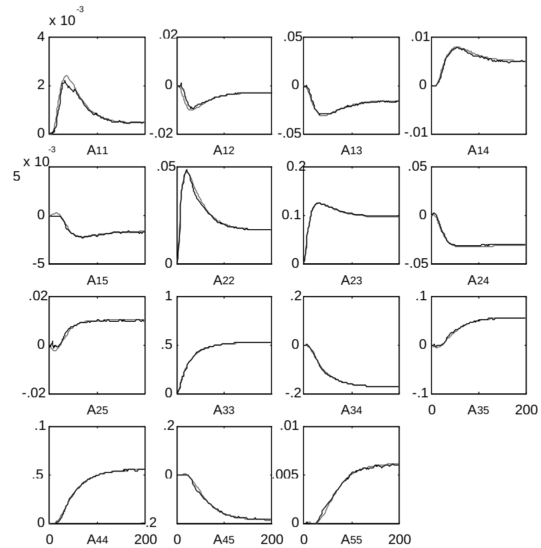

Fig. 5. Simulation and theoretical predictions of time evolution for selected elements $A_{ij} = E[\mathbf{V}(n-1)\mathbf{V}^T(n-1)]_{ij}$ from example (b)

7. REFERENCES

[1] Kuo S. M. and Morgan D.R., *Active Noise Control Systems: Algorithms and DSP Implementations*, New York: John Wiley, 1996.

[2] Elliot S.J. and Nelson P.A., “Active noise control,” *IEEE Signal Processing Magazine*, pp. 12-35, Oct. 1993.

[3] Douglas S. and Pan W. “Exact expectation analysis of the LMS adaptive filter”. *IEEE Trans. on Signal Processing*, pp. 2863-2871, Dec. 1995.

[4] S.D. Snyder and C.H. Hansen, “The effect of transfer function estimation errors on the Filtered-X LMS algorithm,” *IEEE Trans. on Signal Processing*, pp. 950-953, April 1994.

[5] Tobias O.J., Bermudez J.C.M., Bershad N.J. and Seara R., “Mean weight behavior of the Filtered-X LMS algorithm”, *Proc. ICASSP'98* , pp. 3545-3548, 1998.

[6] Bjarnason E., “Analysis of the Filtered-X LMS algorithm”. *IEEE Trans. on Speech and Audio Processing*, pp.504-514, Nov. 1995.

[7] Elliott S.J., Stothers I.M. and Nelson P.A., “A multiple error LMS algorithm and its application to the active control of sound and vibration,” *IEEE Trans. on Acoustics, Speech, and Signal Processing*, pp. 1423-1434, October 1987.