

# MULTIPLE INVARIANCE SPATIO-TEMPORAL SPECTRAL ESTIMATION

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## ABSTRACT

Simultaneous frequency and direction-of-arrival (DOA) estimation can be formulated as a 2D processing problem or as a set of coupled 1D problems. In narrow-band cases the 2D approach can be used to remove the need for frequency-DOA pairing procedures but the price can be a considerably higher computational cost. Therefore, the 1D approach a viable alternative in many applications. This paper presents an ESPRIT based technique using the multiple 1D approach. It requires few sensors, guarantees identifiability of the parameters and admits several sources on the same frequency or DOA.

## 1. INTRODUCTION

In recent years, a number of high resolution eigenstructure based methods have been developed for the 2D narrow-band source location problem, many of which are based on variants of the ESPRIT algorithm. The ESPRIT family of methods have some well-known desirable features that motivates their popularity; they do not require array calibration, they are search free and are fairly robust against array imperfections. ESPRIT can be applied in frequency-DOA estimation either as a 1D method in a separated approach [1]–[4], to yield separate estimates of the frequencies and DOAs, or as a 2D method on the full problem [5],[6]. In the former case one needs a pairing procedure to find the correct frequency-DOA pairs, and one must also by some means ascertain that the pairing problem has a unique solution. The latter can be achieved e.g. with the marked subspace device [3],[4]. Methods for the full 2D problem, such as the recently developed 2D *unitary* ESPRIT [6], avoid this obstacle but the price paid is a higher numerical cost. Typically, 2D methods require eigendecompositions on systems of at least the size  $d^2$ , where  $d$  is the number of sources, whereas for 1D methods the size of the systems is approximately  $d$ . Since the complexities involved often are  $\mathcal{O}(n^3)$ , where  $n$  is the size of the system, 2D methods can give a considerably higher computational cost. Moreover, in 1D methods large parts of the computations can often be performed in parallel, which is important in real-time applications.

In this paper we describe a novel method for frequency-DOA estimation based on ESPRIT, utilizing the separation approach. It exploits the coupling between the space and time variables in the wave equation efficiently and the number of sensors can therefore be kept very low relative to the number of sources. Moreover, parameter identifiability is guaranteed and it is not sensitive to several sources having the same frequency or DOA.

## 2. SPACE-TIME ARRAY GEOMETRY

We consider a plane problem with  $d$  narrow-band sources present, lying in the far-field of the array. At each snapshot the array samples the wavefield in  $M$  space-time points  $(\mathbf{z}_1, t_1), \dots, (\mathbf{z}_M, t_M) \in \mathbb{R}^2 \times \mathbb{R}$ . Assuming identical sensors, the complex output  $\mathbf{y} \in \mathbb{C}^M$  of the array can for each snapshot be represented on the standard form  $\mathbf{y} = \mathbf{A}\mathbf{s} + \mathbf{n}$ , with the steering vectors  $\mathbf{a}(\omega_k, \boldsymbol{\kappa}_k)$  forming the  $d$  columns of  $\mathbf{A}$  and the signal vector  $\mathbf{s} \in \mathbb{C}^d$ , respectively, given by

$$\begin{aligned} \mathbf{a}(\omega_k, \boldsymbol{\kappa}_k) &= \\ &\gamma_k (1, \exp(j(\boldsymbol{\kappa}_k^T (\mathbf{z}_2 - \mathbf{z}_1)) - \omega_k(t_2 - t_1)), \dots \\ &\quad \dots, \exp(j(\boldsymbol{\kappa}_k^T (\mathbf{z}_M - \mathbf{z}_1)) - \omega_k(t_M - t_1)))^T, \\ \mathbf{s} &= (\alpha_1 \exp(j(\boldsymbol{\kappa}_1^T \mathbf{z}_1 - \omega_1 t_1)), \dots \\ &\quad \dots, \alpha_d \exp(j(\boldsymbol{\kappa}_d^T \mathbf{z}_1 - \omega_d t_1)))^T, \end{aligned}$$

and  $\mathbf{n} \in \mathbb{C}^M$  is the noise vector. Here,  $(\omega_k, \boldsymbol{\kappa}_k) \in (0, \infty) \times \mathbb{R}^2$  is the frequency-wavenumber pair of the  $k$ :th signal,  $\gamma_k$  is the corresponding (complex) array gain and  $\alpha_k$  the (complex) source strength (which is random between snapshots). The speed of propagation in the medium is  $c$ , hence  $\|\boldsymbol{\kappa}_k\| = \omega_k/c$ . The frequencies take  $r \leq d$  distinct values and for any given frequency there are at most  $q \leq d - r + 1$  associated wavenumbers. It is further assumed that the array has the following multiple invariance structure. There are four subarrays, indexed by  $\ell = 0, 1, 2, 3$ , such that for some  $N, p$  the  $Np$  space-time points  $(\mathbf{z}_1^{(\ell)}, t_1^{(\ell)}), \dots, (\mathbf{z}_{Np}^{(\ell)}, t_{Np}^{(\ell)})$  sampled by the  $\ell$ :th subarray are related to the others as

$$\begin{aligned} \mathbf{z}_n^{(1)} &= \mathbf{z}_n^{(0)}, & \mathbf{z}_n^{(2)} &= \mathbf{z}_n^{(0)} + \boldsymbol{\Delta}, & \mathbf{z}_n^{(3)} &= \mathbf{z}_n^{(2)}, \\ t_n^{(1)} &= t_n^{(0)} + \tau, & t_n^{(2)} &= t_n^{(0)}, & t_n^{(3)} &= t_n^{(1)}, \end{aligned} \quad (1)$$

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for some nonzero  $\Delta \in \mathbb{R}^2$ ,  $\tau \in (0, \infty)$ . The most compact form of this geometry is depicted schematically in fig. 1. Directions (angles) of arrival  $\theta_k$  are defined relative to the

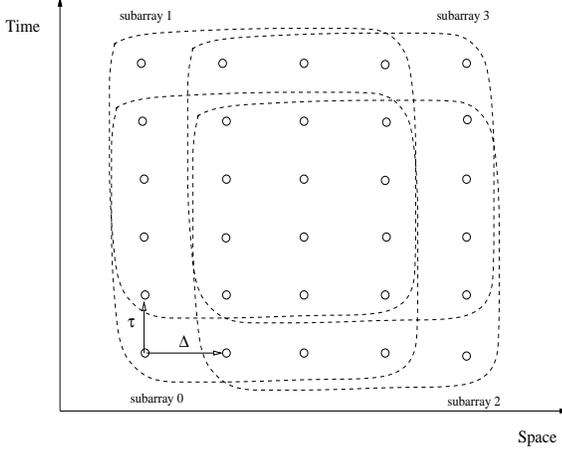


Figure 1: An example of the space-time geometry in (1).

displacement vector  $\Delta$  by  $\cos(\theta_k) = \kappa_k^T \Delta / (\|\kappa_k\| \|\Delta\|)$ , where we assume that all wavenumbers  $\kappa_k$  point into one of the halfplanes separated by a line through  $\Delta$ .

The array geometry in (1) is physically realized by attaching a tapped delay line of  $p$  taps to each of  $N$  equidistantly positioned sensors. With this space-time geometry, the phase shift  $\varphi_k^{(m)}$  experienced by the  $k$ :th signal between the space-time points  $(\mathbf{z}_n^{(0)}, t_n^{(0)})$  and  $(\mathbf{z}_n^{(m)}, t_n^{(m)})$  for  $m = 1, 2, 3$  is independent of  $n$  and given by

$$\begin{aligned} \varphi_k^{(1)} &= -\omega_k \tau, \\ \varphi_k^{(2)} &= \omega_k \frac{\|\Delta\|}{c} \cos(\theta_k), \\ \varphi_k^{(3)} &= \omega_k \left( \frac{\|\Delta\|}{c} \cos(\theta_k) - \tau \right), \end{aligned} \quad (2)$$

where, in order to avoid ambiguities, we assume that  $\varphi_k^{(m)} \in [-\pi, \pi)$  for all  $k, m$ . Let  $\mathbf{y}^{(\ell)}, \mathbf{n}^{(\ell)} \in \mathbb{C}^{Np}$  denote, respectively, the output and noise vector for the  $\ell$ :th subarray. Then,

$$\mathbf{y}^{(0)} = \mathbf{A}^{(0)} \mathbf{s} + \mathbf{n}^{(0)}, \quad \mathbf{y}^{(m)} = \mathbf{A}^{(0)} \Phi^{(m)} \mathbf{s} + \mathbf{n}^{(m)}, \quad (3)$$

where  $\mathbf{A}^{(0)} \in \mathbb{C}^{Np \times d}$  is the matrix of steering vectors for subarray number 0 and  $\Phi^{(m)}$  is the phase factor matrix

$$\Phi^{(m)} = \text{diag}(\exp(j\varphi_1^{(m)}), \dots, \exp(j\varphi_d^{(m)})). \quad (4)$$

### 3. PARAMETER IDENTIFIABILITY

If the noise covariance matrix is known and  $\mathbf{A}^{(0)}, \text{Ess}^H$  both have full rank the (unordered) phase shifts  $\varphi_k^{(m)}$  can be straightforwardly estimated from array output covariance

data using 1D ESPRIT applied to (3), provided  $N \geq d$ . Thus, given that  $N \geq d$  and  $\text{Ess}^H$  has full rank the two remaining issues that have to be solved before the frequency-DOA pairs can be successfully estimated are the rank of  $\mathbf{A}^{(0)}$  and the question of frequency-DOA ambiguity. However, both these issues can be readily resolved given the array geometry in (1). Let  $\phi^{(m)} = (\varphi_1^{(m)}, \dots, \varphi_d^{(m)})^T$ , define  $\tau \in \mathbb{R}^d$  by  $\tau = (\tau, \dots, \tau)^T$  and let  $f : [0, \pi]^d \rightarrow [-1, 1]^d$  be the function that gives the cosine values of the components of a vector. Finally, put  $g = (\|\Delta\|/c)f$  and let  $\odot$  denote the componentwise product between vectors.

**Proposition 1.** *Suppose that no two distinct components of  $\phi^{(1)}, \phi^{(2)}$  and  $\phi^{(3)}$  have the same values.<sup>1</sup> Then for any three  $d \times d$  permutation matrices  $\mathbf{P}_0^{(1)}, \mathbf{P}_0^{(2)}$  and  $\mathbf{P}_0^{(3)}$  there exist three  $d \times d$  permutation matrices  $\mathbf{P}^{(1)}, \mathbf{P}^{(2)}$  and  $\mathbf{P}^{(3)}$ , which are unique up to a common  $d \times d$  row permutation factor  $\mathbf{Q}$ , such that the system of equations*

$$\begin{cases} \mathbf{P}^{(1)} \mathbf{P}_0^{(1)} \phi^{(1)} &= -\xi \odot \tau \\ \mathbf{P}^{(2)} \mathbf{P}_0^{(2)} \phi^{(2)} &= \xi \odot g(\eta) \\ \mathbf{P}^{(3)} \mathbf{P}_0^{(3)} \phi^{(3)} &= \xi \odot (g(\eta) - \tau) \end{cases}, \quad (5)$$

in  $\xi \in (0, \infty)^d, \eta \in [0, \pi]^d$  has a solution. The permutation matrices  $\mathbf{P}^{(1)}, \mathbf{P}^{(2)}, \mathbf{P}^{(3)}$  are given by  $\mathbf{P}^{(m)} = \mathbf{Q}(\mathbf{P}_0^{(m)})^{-1}$  and the corresponding solution, which is unique, is given by  $\hat{\xi} = \mathbf{Q}\omega, \hat{\eta} = \mathbf{Q}\theta$ .

The proof is straightforward and omitted. The result says that given unordered phase shifts  $(\mathbf{P}_0^{(m)} \phi^{(m)})$ , there is only one reordering of them that makes the system in (5) solvable and the corresponding solution gives the correct frequency-DOA pairs, but possibly in a different order (as expressed by  $\mathbf{Q}$ ). In other words, the array geometry in (1) guarantees that no frequency-DOA ambiguity can exist (generically). As for the rank of  $\mathbf{A}^{(0)}$  we can use a result on Vandermonde matrices. Let  $\mathcal{V}^{(p,d)} \subset \mathbb{C}^{p \times d}$  be the set of ‘‘unit Vandermonde’’ matrices with the  $k$ :th row given by  $((v_1)^{k-1}, \dots, (v_d)^{k-1}) \in \mathbb{C}^{1 \times d}, |v_1| = \dots = |v_d| = 1$ .

**Proposition 2.** *Let  $\mathbf{V} \in \mathcal{V}^{(p,d)}$  be a unit Vandermonde matrix with no value in the second row  $(v_1, \dots, v_d)$  occurring with larger multiplicity than  $q$  and let  $\mathbf{D} \in \mathbb{C}^{d \times d}$  be a diagonal matrix with distinct nonzero entries on the diagonal. Then, if  $p \geq d$  and  $N \geq q$  the matrix  $\mathbf{W} \in \mathbb{C}^{Np \times d}$  given by*

$$\mathbf{W} = \begin{pmatrix} \mathbf{V} \\ \mathbf{VD} \\ \vdots \\ \mathbf{VD}^{N-1} \end{pmatrix} \quad (6)$$

has full rank ( $= d$ ).

<sup>1</sup>This holds with probability one under mild assumptions.

The proof is given in appendix. The result can be directly applied to the rank problem of  $\mathbf{A}^{(0)}$  since this matrix can be written as  $\mathbf{A}^{(0)} = \mathbf{W}\mathbf{G}$ , where  $\mathbf{W}$  has the structure in (6) with  $\mathbf{D}$  of the form (4) and  $\mathbf{G}$  is a diagonal nonsingular matrix.<sup>2</sup> It follows that when the array has the geometry (1) with  $p \geq d$  and  $N \geq q$  the subarray response matrix  $\mathbf{A}^{(0)}$  has generically full rank. In other words, whenever the taps  $p$  attached to each of the sensors in the array is at least as large as the number of signals  $d$ , and the space points  $N$  in each subarray is at least as large as the largest number of DOAs at a single frequency  $q$ , the subarray response matrix  $\mathbf{A}^{(0)}$  can be expected to have full rank.

What remains to be addressed is how to actually perform the frequency-DOA pairing. By the geometry of the array the phase-difference vectors satisfy the linear relation

$$\phi^{(1)} + \phi^{(2)} = \phi^{(3)}, \quad (7)$$

and this suggests the following simple procedure, given the unordered components of  $\phi^{(m)}$ : 1) List the components of  $\phi^{(m)}$  in an input table and set up an empty output table. 2) Find indices  $k_m$  such that  $\phi_{k_1}^{(1)} + \phi_{k_2}^{(2)} = \phi_{k_3}^{(3)}$ . 3) Store the ordered triple  $(\phi_{k_1}^{(1)}, \phi_{k_2}^{(2)}, \phi_{k_3}^{(3)})$  in the output table and remove  $\phi_{k_1}^{(1)}, \phi_{k_2}^{(2)}, \phi_{k_3}^{(3)}$  from the input table. 4) Proceed from 2) until the input table is exhausted. 5) Finally, calculate the frequency-DOA pairs using (2). In practice, relation (7) is never exactly fulfilled and one has to complement the pairing procedure with some sort of error criterion and fitting procedure to determine when a best match resembling (7) is found (such as a mean square error criterion, which is what is used in the simulations below).

#### 4. SIMULATIONS

In all examples the signal magnitudes  $|\alpha_k|$  are constant and equal, and the phases  $\arg(\alpha_k)$  are random with statistically independent between signals and snapshots) uniformly distributed phases. The noise vectors have independent components with independent Gaussian real and imaginary parts. Estimates of the array output correlation matrix are calculated by standard averaging of outer products of data vectors and the ‘noise cleaning’ process to obtain the signal output correlation matrix from the array output correlation matrix is done by subtraction of a multiple of the identity matrix. The minimal dimension of the composite array output correlation matrix has always been used (i.e. matrix sizes have been reduced due to overlapping space-time points wherever possible). The signal-to-noise ratio (SNR) is defined as  $\text{SNR} = 10 \log_{10}(d E|s_1|^2 / E|n_1|^2)$ , where  $s_1, n_1$  are the first component of the signal and noise vectors, respectively.

<sup>2</sup>Also in the condition that  $\mathbf{D}$  in this case has distinct components holds with probability one under mild assumptions.

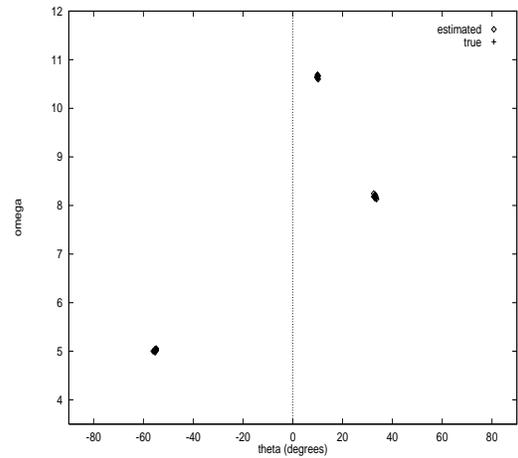


Figure 2: Example A, 200 snapshots, SNR = 20dB.

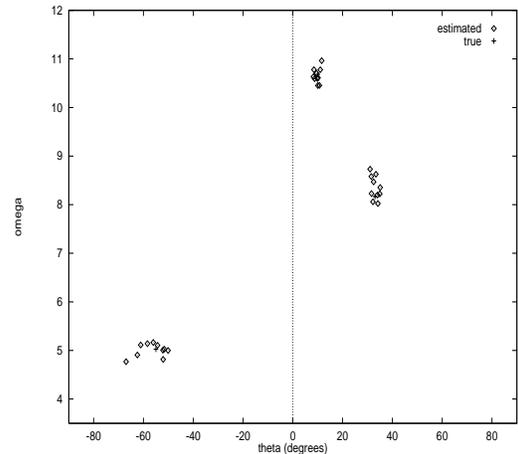


Figure 3: Example A, 200 snapshots, SNR = 6dB.

DOAs are measured in degrees, frequencies are given dimensionless in multiples of  $2\pi$  and estimates of the phase differences  $\phi^{(m)}$  are computed by TLS-ESPRIT.

*Example A.* Three sources present, located in the  $\omega, \theta$ -plane at  $(0.8 \cdot 2\pi, -55^\circ)$ ,  $(1.7 \cdot 2\pi, 10^\circ)$  and  $(1.3 \cdot 2\pi, 33^\circ)$ , respectively. The number of taps  $p$  and sensors  $N$  in each subarray are  $p = 5, N = 2$ , the value of  $\tau$  is 0.25, which is a quarter of a period at nominal frequency 1, and  $\|\Delta\|$  equals a quarter of wavelength at nominal frequency. Scatter plots obtained from 10 runs of 200 samples each are given in figs 2,3 for SNR=20dB and SNR=6dB, respectively.

*Example B.* Same setting as example A but with one additional source at  $(1.3 \cdot 2\pi, -30^\circ)$  (thus  $q = 2$ ) and  $p = 6, N = 3$ . Scatter plots for SNR=20dB and SNR=6dB, respectively, are shown in figs 4,5.

#### 5. CONCLUSIONS

The method presented can be viewed as an algebraic frequency separation of the frequency-DOA estimation prob-

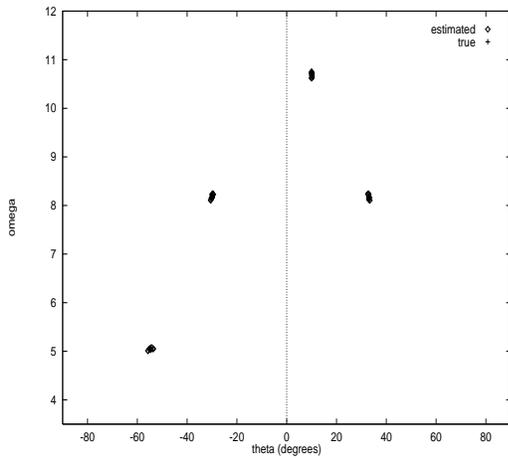


Figure 4: Example B, 200 snapshots, SNR = 20dB.

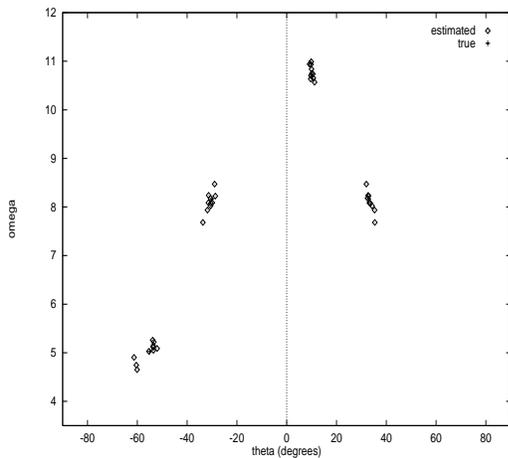


Figure 5: Example B, 200 snapshots, SNR = 6dB.

lem, so that the number of sensors needed is not determined by the total number of signals but the *maximal number of signals per frequency* since the requirement is  $N \geq q$ . (In coherent cases spatio-temporal smoothing can be employed). With maximal space-time overlap, the largest eigensystem to be solved is of the size  $Np$ , where  $N$  is of the order  $q$  and  $p$  is of the order  $d$ . For small  $q$  this size is only slightly larger than  $d$ , which often is considerably smaller than  $d^2$ . In scenarios such as passive sonar, where  $q$  is generally small, the resulting low requirements in terms of sensors and computations should make the method attractive.

## 6. APPENDIX

The only case really needed to prove is  $r < d$  (i.e.  $q > 1$ ) and  $N \geq q \geq 2$ . Assume without loss of generality that the elements in  $(v_1, \dots, v_d)$  are ordered according to increasing complex argument in  $[-\pi, \pi)$ . Then the matrix  $\mathbf{V}$  can be written  $\mathbf{V} = (\mathbf{V}_1, \dots, \mathbf{V}_r)$  where matrix  $\mathbf{V}_k$  consists of  $n_k$  identical columns. By picking one column each from these

matrices we could form a Vandermonde matrix with distinct generating elements, which has full rank, so  $\text{rank}(\mathbf{V}) = r$  and  $\mathcal{R}(\mathbf{V}) = \bigoplus_{k=1}^r \mathcal{R}(\mathbf{V}_k)$ . Now, if  $\mathbf{W}\mathbf{u} = \mathbf{0}$  we have

$$\mathbf{0} = \mathbf{V}_k \mathbf{u}_k = \mathbf{V}_k \mathbf{D}_k \mathbf{u}_k = \dots = \mathbf{V}_k \mathbf{D}_k^{N-1} \mathbf{u}_k, \quad (8)$$

for  $k = 1, \dots, r$ , where  $\mathbf{D}_k \in \mathbb{C}^{n_k \times n_k}$  is the submatrix of  $\mathbf{D}$  corresponding to  $\mathbf{V}_k$  in the product  $\mathbf{V}\mathbf{D}$  and  $\mathbf{u}_k \in \mathbb{C}^{n_k}$  is the subvector of  $\mathbf{u}$  corresponding to  $\mathbf{D}_k$  in the product  $\mathbf{D}\mathbf{u}$ . The equalities in (8) can be written compactly as

$$\mathbf{0} = \mathbf{V}_k \mathbf{U}_k (\mathbf{1}, \mathbf{d}_k^{(1)}, \dots, \mathbf{d}_k^{(N-1)}), \quad k = 1, \dots, r, \quad (9)$$

where  $\mathbf{U}_k \in \mathbb{C}^{n_k \times n_k}$  is a diagonal matrix with the components of  $\mathbf{u}_k$  along the diagonal and  $\mathbf{d}_k^{(\nu)} \in \mathbb{C}^{n_k}$  is the vector obtained by stacking the entries on the diagonal of  $\mathbf{D}_k^\nu$  on top of each other. If  $\mathbf{u}_k \neq \mathbf{0}$  we have  $\text{rank}(\mathbf{V}_k \mathbf{U}_k) = 1$  and since  $(\mathbf{1}, \mathbf{d}_k^{(1)}, \dots, \mathbf{d}_k^{(N-1)})$  is a (transposed) Vandermonde matrix with distinct generating elements it has full rank so (9) implies  $n_k - 1 \geq \min\{N, n_k\}$ , i.e.  $N \leq n_k - 1$ . On the other hand,  $q \geq n_k$  and  $N \geq q$  so  $N \geq n_k$ , which is a contradiction. Hence,  $\mathbf{u}_k = \mathbf{0}$  for all  $k$  and  $\mathcal{N}(\mathbf{V}) = \{\mathbf{0}\}$ .

## 7. REFERENCES

- [1] M.D. Zoltowski and D. Stavrinos, "Sensor Array Signal Processing Via a Procrustes Rotations Based Eigenanalysis of the ESPRIT Data Pencil," *IEEE Trans. Acoust. Speech Signal Processing*, Vol. 37, No. 6, pp. 832–860, June 1989.
- [2] A.J. van den Veen, P.B. Ober and E.F. Deprettere, "Azimuth and Elevation Computation in High Resolution DOA Estimation," *IEEE Trans. Signal Processing*, Vol. 40, No. 7, pp. 1828–1832, July 1992.
- [3] Y.-H. Chen and C.-H. Chen, "Direction-of-Arrival and Frequency Estimations for Narrowband Sources Using Two Single Rotation Invariance Algorithms with the Marked Subspace," *IEE Proceedings-F*, Vol. 139, No. 4, pp. 297–300, Aug. 1992.
- [4] V.S. Kedia and B. Chandna, "A New Algorithm for 2-D DOA Estimation," *Signal Processing*, Vol. 60, No. 3, pp. 325–332, Aug. 1997.
- [5] F.A. Sakaya and M.H. Hayes, "Estimating 2-D DOA Angles Using Nonlinear Array Configurations," *IEEE Trans. Signal Processing*, Vol. 43, No. 9, pp. 2212–2216, Sept. 1995.
- [6] M.D. Zoltowski, M. Hardt and C.P. Mathews, "Closed-Form 2-D Angle Estimation with Rectangular Arrays in Element Space or Beamspace via Unitary ESPRIT," *IEEE Trans. Signal Processing*, Vol. 44, No. 2, pp. 316–328, Feb. 1996.