CONDITIONAL MAXIMUM LIKELIHOOD TIMING RECOVERY

Jaume Riba and Gregori Vazquez

Department of Signal Theory and Communications. Universitat Politècnica de Catalunya.
Campus Nord, Ed. D5, Jordi Girona 1 i 3, 08034, Barcelona (Spain).
Email: jriba,gregori@gps.tsc.upc.es

ABSTRACT

The Conditional Maximum Likelihood (CML) Principle, well known in the context of sensor array processing, is applied to the problem of timing recovery. A new self-noise free CML-based timing error detector is derived. Additionally, a new (Conditional) Cramer-Rao Bound (CRB) for timing estimation is obtained, which is more accurate than the extensively used *modified* CRB (MCRB).

1. INTRODUCTION

One of the fundamental tasks of a digital receiver is the estimation of the symbol timing directly from the received data. The recent book by Mengali and D'Andrea [1] as well as the F. Gardner's report [2] constitute excellent references for this topic of synchronization. Timing recovery algorithms are typically categorized in Decision-Directed (DD) and Non-Data_Aided (NDA) methods. While DD schemes offer better tracking performance, NDA methods are preferred when the decisions are not available or not reliable. NDA algorithms offer the additional advantage of being phase-independent, thus avoiding spurious locks and prolonged acquisitions caused by complex iteractions between phase and timing correction algorithms.

Maximum Likelihood (ML) estimation techniques offer a systematic and conceptually simple guide to derive synchronization algorithms which provide optimum or near optimum performance against noise. While the application of the ML principle is straightforward for the derivation of DD algorithms, mathematical limitations arise, however, in the derivation of NDA methods. Then, ML-oriented approaches have been employed in the literature by resorting to approximations and heuristic reasoning. On the other hand, completely *ad hoc* methods have also been brought out which offer a significant simplification of the implementation complexity.

Insuperable mathematical problems also arise in the computation of the Cramer-Rao Bound (CRB), which establishes a fundamental lower limit to the variance of any unbiased estimator. A more manageable performance limit is the *modified* CRB (MCRB) proposed by d'Andrea et al. [3]. This bound is generally lower than (at most equal to) the true CRB, and it is difficult to know in advance whether the MCRB is tight enough for use in practical applications.

In this contribution we adopt the Conditional ML (CML) approach which has been widely applied to the problem of Direction-Of-Arrival (DOA) estimation using sensor arrays (see Stoica and Nehorai paper [4] and references therein). The application of this principle to the frequency estimation problem can be found in [5], and its general application to synchronization problems was proposed in [6]. When adopting the conditional model, the data symbols, which play the same role as the sources in the DOA context, are modelled as *deterministic unknown* parameters. It is shown that the application of the CML principle does not need any additional approximation nor heuristic reasoning, and leads to a timing error detector structure which does not exhibit self-noise. We also derive the (asymptotically) true CRB for timing recovery under the conditional assumption.

2. DISCRETE-TIME SIGNAL MODEL

We assume that the received waveform has a complex envelope:

$$r(t) = s(t) + w(t) \tag{1}$$

where $s(t)$ is the information-bearing signal and $w(t)$ represents complex-valued white Gaussian noise with two-sided power spectral density $2N_o$. The signal $s(t)$ is modelled as follows:

$$s(t) = Ae^{j\theta} \sum_{i=0}^{L-1} c_i g(t - iT - \tau) \tag{2}$$

where τ is the timing parameter to be estimated, θ is the signal phase, A is the signal amplitude, T is the symbol spacing, $\{c_i\}$ are complex-valued symbols, L is the number of symbols and $g(t)$ is the (real-valued) signalling pulse. The set of unknown, undesired parameters includes the signal amplitude, the signal phase and the data, and it is denoted by the following vector:

$$\mathbf{x} = A\mathbf{c}e^{j\theta} \tag{3}$$

where the data symbol vector is:

$$\mathbf{c} = [c_0 \cdots c_{L-1}]^T \tag{4}$$

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In order to apply the theory developed for sensor array processing, we derive in the sequel a discrete-time signal model, although the results obtained are general, irrespective of whether an analog or digital receiver is used. To this end we chose a sampling frequency of $f_s = 1/T_s = K/T$, where K is the minimum integer that guarantees the absence of aliasing. In these circumstances, the performance of the resulting estimator should not be dependent on the value of K. After an ideal antialiasing filtering of bandwidth $f_s/2$, (1) and (2) can be written as follows:

$$\mathbf{r} = [r(0) \cdots r((M-1)T_s)]^T = \mathbf{A}_\tau \mathbf{x} + \mathbf{w} \qquad (5)$$

where M is the number of non-zero samples of $r(t)$, which depends on the effective length of the signalling pulse, and:

$$\begin{aligned}
\mathbf{A}_\tau &= [\mathbf{a}_0(\tau) \cdots \mathbf{a}_{L-1}(\tau)] \\
\mathbf{a}_i(\tau) &= [g(-iT-\tau), g(T_s - iT - \tau) \cdots \\
&\quad g((M-1)T_s - iT - \tau)]^T \\
\mathbf{w} &= [w_0 \cdots w_{M-1}]^T \\
\mathbf{C_w} &= E\left[\mathbf{w}\mathbf{w}^H\right] = \sigma^2 \mathbf{I} = 2N_o f_s \mathbf{I}
\end{aligned}$$

3. CML-BASED TIMING ERROR DETECTOR

The signal model (5) is widely used in the context of sensor array processing (see for instance [4]), were $\mathbf{x}$ is the signal source vector, $\mathbf{r}$ is the snapshot and $\mathbf{A}$ is the DOA-dependent transfer matrix. The only difference is that, in the timing estimation problem, the whole transfer matrix $\mathbf{A}$ is parametrized solely by the timing parameter τ. In the presence of AWGN, the CML function for the estimation of τ can be expressed as [4]:

$$L_c(\mathbf{r}|\tau) = tr\left[\mathbf{P}_{A_\tau}^{\perp} \widehat{\mathbf{R}}\right] = \mathbf{r}^H \mathbf{P}_{A_\tau}^{\perp} \mathbf{r} \qquad (7)$$

where $\mathbf{P}_A^{\perp} = \mathbf{I} - \mathbf{A}\mathbf{A}^{\sharp}$ is the projector onto the orthogonal signal subspace and $\mathbf{A}^{\sharp} = \left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{A}^H$ is the pseudoinverse of matrix $\mathbf{A}$. The CML NDA timing estimator is defined as the minimizer of (7). To derive a CML timing error detector we need to compute the derivative of the CML function with respect to τ, and use it as an error signal to drive the function $L_c(\mathbf{r}|\tau)$ toward its minimum. The CML gradient has been obtained by Viberg, Ottersten and Kailath [7] within the more general context of sensor array processing. For the problem of timing estimation, the general gradient expression can be manipulated to yield:

$$g_c(\overline{\tau}) = \frac{d}{d\tau} L_c(\mathbf{r}|\tau) = -2\,\mathrm{Re}\left[\left(\mathbf{r}^H \mathbf{P}_{A_\tau}^{\perp} \mathbf{D}_\tau\right)\left(\mathbf{A}_\tau^{\#}\mathbf{r}\right)\right] \qquad (8)$$

It is seen that the gradient is estimated by measuring the crosscorrelation at the output of two filters, $\mathbf{A}_\tau^{\#}$ and $\mathbf{D}_\tau^H \mathbf{P}_{A_\tau}^{\perp}$ applied to signal vector $\mathbf{r}$. To obtain a practical TED we are interested in the asymptotic form of these two matrices as the number of symbols L approaches infinity. For large L, the adjacent central rows of $\mathbf{A}_\tau^{\#}$ differ asymptotically in a time shift equal to a symbol interval, and they correspond to the impulse response of a zero forcer. The same asymptotic behavior is found for matrix $\mathbf{D}_\tau^H \mathbf{P}_{A_\tau}^{\perp}$, whose central rows converge to a specific shape. As a consequence, the matrix-by-vector operations $\mathbf{A}_\tau^{\#}\mathbf{r}$ and $\mathbf{D}_\tau^H \mathbf{P}_{A_\tau}^{\perp}\mathbf{r}$ in (8) can be viewed as time-invariant filters whose outputs are decimated at one sample per symbol, and then multiplied in a symbol-by-symbol basis to yield the timing error indication. The impulse response of these filters is computed as follows:

$$\begin{aligned}
g_c(t) &\leftarrow \text{central row} \lim_{L\to\infty} \mathbf{A}_\tau^{\#} \\
d_c(t) &\leftarrow \text{central row} \lim_{L\to\infty} \mathbf{D}_\tau^H \mathbf{P}_{A_\tau}^{\perp} \qquad (9)
\end{aligned}$$

The final structure of the asymptotic CML TED is shown in figure (1).

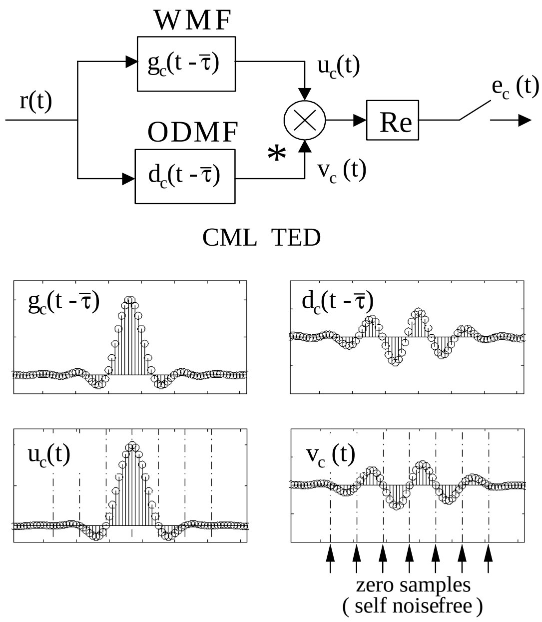

Figure 1: Structure of the the new CML TED and the illustration of the absence of self noise.

The obtained CML-TED is similar in structure to the classical ML-oriented TED [1] derived under the unconditional model assumption, i.e., by assuming the symbols are independent random variables of a density function dependent on the signal constellation, and modelling the signal phase as a uniform random variable. The only difference with the classical ML-oriented TED structure is in the definition of the two branch filters. These filters will be referred to as Whitened Matched Filter (WMF) ($g_c(t)$) and Orthogonal Derivative Matched Filter (ODMF) ($d_c(t)$). The main advantage of the new solution is that, in contrast to the Derivative Matched Filter (DMF) used in the classical structure, the ODMF does not generate self noise because its output in the noiseless case is $\mathbf{D}_\tau^H \mathbf{P}_{A_\tau}^{\perp} \mathbf{A}_\tau \mathbf{x} = \mathbf{0}$ in the absence of timing error, as illustrated in figure 1 by the zero strobe samples at the ODMF output.

4. TRUE CRB FOR TIMING RECOVERY

Under the unconditional model assumption, the derivation of the CRB poses insuperable obstacles. An alternative bound is the *modified* CRB which yields [3]:

$$\mathrm{MCRB}\left(\tau\right)=\frac{1}{8\pi^{2}L\xi}\frac{T^{2}}{E_{s}/N_{o}} \tag{10}$$

where ξ is an adimensional coefficient depending on the shape of $g(t)$:

$$\xi=\frac{\int_{-\infty}^{\infty}T^{2}f^{2}\left|G(f)\right|^{2}df}{\int_{-\infty}^{\infty}\left|G(f)\right|^{2}df} \tag{11}$$

For comparison purposes, it will be useful to express the previous coefficient in the discrete-time domain using the Parseval theorem:

$$\xi=\frac{T^{2}}{4\pi^{2}E_{g}}T_{s}\left\|\mathbf{d}_{0}(\tau)\right\|^{2} \tag{12}$$

It is demonstrated in [3] that the MCRB is generally lower than (at most equal to) the true CRB:

$$\mathrm{CRB}\left(\tau\right)\geq\mathrm{MCRB}\left(\tau\right) \tag{13}$$

In the sequel, we derive a new bound under the conditional model assumption, by using the high amount of research effort in the field of array processing theory. In the context of DOA estimation using sensor arrays, Stoica and Nehorai [4] derived the *Conditional* Cramer-Rao bound (CRB_c), which for the problem at hand can be expressed as:

$$\mathrm{CRB}_{c}\left(\tau\right)=\frac{\sigma^{2}}{2\mathbf{x}^{H}\mathbf{D}_{\tau}^{H}\mathbf{P}_{A_{\tau}}^{\perp}\mathbf{D}_{\tau}\mathbf{x}} \tag{14}$$

It is noted that the conditional CRB depends on the specific symbol sequence $\mathbf{x}$. This may be useful for evaluating the ultimate performance of timing estimators designed for burst mode applications, when an specific finite-length preamble is used for initial timing recovery. However, in most cases we are interested in the best performance that can be attained by a timing estimator operating in continuous mode. In that case, the statistical properties of the data should play a fundamental role. To obtain an asymptotic performance bound we note that the denominator of (14) is a consistent estimate of the energy of $\mathbf{x}$ with respect to the matrix $\mathbf{D}_{\tau}^{H}\mathbf{P}_{A_{\tau}}^{\perp}\mathbf{D}_{\tau}$. Therefore the asymptotic conditional CRB is given by [4]:

$$\mathrm{CRB}_{c}^{as}\left(\tau\right)=\frac{\sigma^{2}}{2\mathrm{tr}\left(\mathbf{D}_{\tau}^{H}\mathbf{P}_{\mathbf{A}_{\tau}}^{\perp}\mathbf{D}_{\tau}\mathbf{\Gamma}\right)} \tag{15}$$

where:

$$\mathbf{\Gamma}=E_{\mathbf{x}}\left[\mathbf{x}\mathbf{x}^{H}\right] \tag{16}$$

is the covariance matrix of the symbols. Under the standard assumption that the symbols are zero-mean independent random variables ($\mathbf{\Gamma}=\sigma_{c}^{2}\mathbf{I}$), we can write:

$$\mathrm{CRB}_{c}^{as}\left(\tau\right)=\frac{\sigma^{2}}{2\sigma_{c}^{2}\sum_{i=0}^{L-1}\left\|\mathbf{P}_{\mathbf{A}_{\tau}}^{\perp}\mathbf{d}_{i}(\tau)\right\|^{2}} \tag{17}$$

After some manipulations we obtain the following expression:

$$\mathrm{CRB}_{c}^{as}\left(\tau\right)=\frac{1}{8\pi^{2}L\xi_{c}}\frac{T^{2}}{E_{s}/N_{o}} \tag{18}$$

where ξ_c is an adimensional coefficient depending also on the shape of $g(t)$:

$$\xi_{c}=\frac{T^{2}}{4\pi^{2}E_{g}}\frac{T_{s}}{L}\sum_{i=0}^{L-1}\left\|\mathbf{P}_{\mathbf{A}_{\tau}}^{\perp}\mathbf{d}_{i}(\tau)\right\|^{2} \tag{19}$$

The significance of the new CRB for the timing estimation problem obtained in (18) is twofold. On the one hand, it holds that:

$$\lambda_{L}=\frac{\xi_{c}}{\xi}\leq1 \tag{20}$$

[1] implying that:

$$\mathrm{CRB}_{c}^{as}\left(\tau\right)\geq\mathrm{MCRB}\left(\tau\right) \tag{21}$$

which means that the new bound is more accurate than the modified CRB. On the other hand, Stoica and Nehorai showed [4] that, although in general the $\mathrm{CRB}_{c}^{as}\left(\tau\right)$ cannot be attained, it converges to the true (unconditional) CRB when the SNR increases or the dimension M of the signal vector $\mathbf{r}$ increases. While in the context of sensor array processing the dimension of M is equal to the number of sensors (and it does not depend on L) in the context of timing estimation, M is the dimension of the signal which increases in proportion with the number of symbols L. For that reason, the new bound derived in (18) converges to the true CRB for large L. Therefore, the coefficient λ_L in (20) for $L\rightarrow\infty$ measures the department between the *modified* CRB and the true CRB.

Figure (2) shows the evolution of λ_L as a function of the *roll-off* parameter for increasing L. It is seen that the most difficult situation for the timing estimation is in the lower range of the *roll-off* parameter. Although this fact is already reflected by the classical coefficient ξ in (12) (which is sensitive to the second order moment of the signal spectrum), the new coefficient $\xi_c=\xi\lambda_L$ in (19) shows a stronger dependence with this parameter. While ξ measures *only* the degree of detectability of a single pulse in noise, ξ_c takes also into account the fact that the L pulses are received with a certain degree of overlapping, which is higher for smaller *roll-off*. In the classical (UML) approach, this fact is not considered due to the heuristic approximations adopted. As a result, the obtained estimator is affected by self noise (non-zero strobe samples at the DMF output) and the associated performance limit (MCRB) is optimistic. We have seen that the CML formulation solves this limitation, making unnecessary to resort to *ad hoc* prefiltering techniques [8] for explicitly cancelling the self noise.

5. SIMULATION RESULTS

Numerical results are presented here to demonstrate the tracking performance of the CML TED compared with the

[1] Note that the only difference between (19) and (12) is a projection operation which will never increase the norm of vectors $\mathbf{d}_i(\tau)$. On the other hand, these norms are all equal: $\left\|\mathbf{d}_{i}(\tau)\right\|^{2}=\left\|\mathbf{d}_{0}(\tau)\right\|^{2}\quad\forall i$ and not dependent on τ.

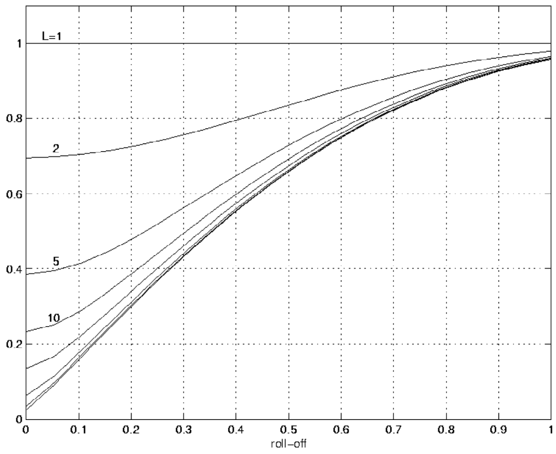

Figure 2: λ_L as a function of the *roll-off* .

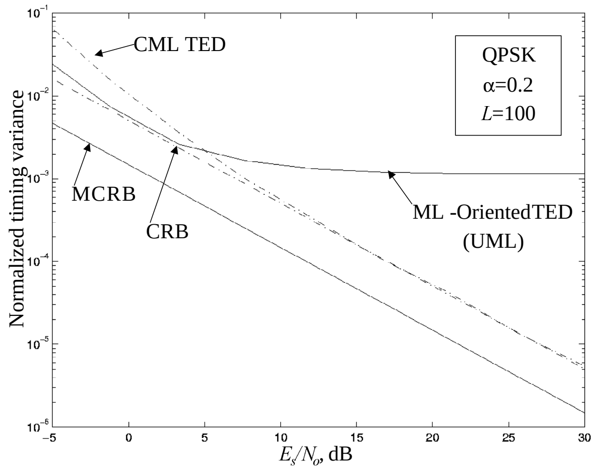

Figure 3: Tracking performance of the CML TED compared to the classical ML-oriented TED. The MCRB and the new CRB are also shown. (*roll-off*=0.2).

classical ML-oriented TED (UML TED). Figure 3 shows the normalized (with respect to T^2) timing variance as a function of E_s/N_o. Modulation is QPSK and the overall channel response is Nyquist with *roll-off* =0.2 respectively. In both cases a loop bandwidth of $5 \cdot 10^{-3}$ is chosen, which corresponds to an effective memory of $L = 100$ symbols.

It is seen that the CML TED attains the CRB at high E_s/N_o while the classical ML-oriented TED (or UML TED) has a floor timing jitter due to self noise. In contrast, the CML TED shows a variance penalty in the lower range of E_s/N_o. This penalty is higher for small excess bandwidth (*roll-off*), which is the case of higher department between the MCRB and the CRB. For different *roll-off* parameters, the department between the CRB and the MCRB is different, according to the factor λ_∞ (see figure 2).

6. CONCLUSIONS

In this paper, the concept of conditional ML and conditional CRB, well known in the context of sensor array processing, have proven useful in timing synchronization. It leads naturally to a timing error detector structure which is free of self-noise, without requiring any approximation nor heuristic approaches. The CML timing error detector has the same structure as that of the ML-oriented estimator, where the matched filter is replaced by the whitened matched filter and the derivative matched filter is replaced by the orthogonal derivative matched filter. The conditional model assumption has also allowed the computation of the true CRB, thus making unnecessary the use of the classical MCRB approximation.

The future work will focus on the extension of the theory to non-linear and staggered modulation formats.

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