# A NEW VIEW OF SHAPE

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## ABSTRACT

A new theory that qualitatively and quantitatively describes shape is presented. Signals are treated as ordered sets and shape as a property of the order. Measures of shape are derived starting from simple two element sets. It is shown that the degree of asymmetry or relative contrast at various scales is a sufficient descriptor of shape. The shape of a signal is a composite of the shape of all possible ordered subsets. A distinction is drawn between statistical and shape measures within a unified theoretical framework which may make it possible to compare diverse pattern recognition approaches in terms of robustness and discrimination power.

## 1. INTRODUCTION

There are a multitude of approaches to pattern recognition in one and higher dimension – statistical, structural or syntactic, and connectionist or artificial neural methods. There are numerous feature extraction methods to extract shape. Shape is extracted from contours [1], symmetry [2], shading [3], moments [4], Fourier [5] and higher order spectral representations [6,7]. One finds reference to global and local shape in the literature without any formal definition. Multiresolution techniques, such as scale-space [8] and wavelet transforms [9,10], attempt to capture shape information at various degrees of localisation. In the absence of a formal definition and theoretical framework, it is difficult to compare diverse methods effectively. Most comparisons are based on classification accuracies obtained on common data - they reveal little about the discriminating power and robustness of the techniques in general. An attempt at formally defining shape and placing it in a mathematical framework that may permit systematic comparisons is presented here.

### 1.1 Axioms Corollaries and Definitions

**Axiom A1:** A null set $\phi$ has no shape.

**Axiom A2:** A set of one element has one shape regardless of the value of the element. It can have many possible values. A value is an intrinsic property of a single element, such as a number, or colour, or quality.

**Axiom A3:** An ordered set of two or more elements has more than one shape.

**Definition D1:** Shape is a property of the order of an ordered set and all its possible ordered subsets. For example, the shape of A = {1,2,3} is a property of {1,2,3}, {1,2}, {1,3}, {2,3}, {1}, {2}, {3} and $\phi$.

**Corollary D1.1:** If the order of values in a set is changed, the shape changes. For example, A = {1,2,3} and B = {2,3,1} have different shapes. However, they may have similar shaped subsets such as {2,3}.

**Corollary D1.2:** A value is itself an element of a set of possible values called the range. The range may or may not have an order within it. For example, a set of integers is ordered, a set of colours is not ordered, a set of colours represented by numbers has an order imposed on its elements.

**Corollary D1.3:** A set of constants has only one shape because all orderings of the elements are identical.

A measure of shape must measure "changes" in the intrinsic properties or values of elements when the order of elements is changed in a prescribed manner. Without loss of generality this change in order can be chosen as the reversal because any other change can be specified in terms of reversals of the order of elements in subsets of the set. In order to measure the "change" operations must be defined on the values of the elements. For the remainder of the discussion, numerical values will be assumed. The concepts described can be extended to other values with appropriate definitions of operations on these values.

### 1.2 Shape of a 2-element Set

**Definition D2:** For a two-element set $\{x(1), x(2)\}$ of real numbers, that are elements of a group with addition and multiplication as defined operations, define the shape to be

$$\psi_2 = \frac{x(1) - x(2)}{x(1) + x(2)} \qquad (1)$$

This measure is consistent with Weber's law for perception of contrast that states that the smallest perceptible change in brightness is proportional to the background brightness. Henceforth, the value of an element will be referred to as the brightness or intensity. An order from the "darkest" to the "brightest" is assumed on these values, represented by real numbers.

The "zero" of values may be taken as the "darkest" or least perceptible or imperceptible value in describing absolute perception by an ideal sensor. However, shape is a relative property and describes a change in value. These changes must be described with respect to a reference for perception. This reference could be the background brightness or adaptation level of the sensor. For example, a pattern on grey paper is visible if it is darker or lighter than the paper. The temperature of an object can be perceived by touch if it is colder or warmer than the temperature of the human body.

**Corollary D2.1:** A two-element set of real numbers has an infinite number of possible shapes. They belong to three major categories - left asymmetric if the shape is negative, right asymmetric if the shape is positive, and symmetric if the shape is zero.

**Corollary D2.2:** The shape of a two-element set is unchanged by multiplication of the element values by a constant. This property is desirable in an extension of the definition of shape to larger ordered sets.
**Corollary D2.3:** The shape of a set of zeroes is indeterminate. This is consistent with the interpretation of zero as an imperceptible value. There is no shape for a completely imperceptible ordered set.
**Corollary D2.4:** Shape is unchanged by a negation of the intensities of all elements. A negative value is interpreted as a perception of the opposite quality - that is, if positive values are brighter than the reference, negative values are darker. An increase in brightness is equivalent to a decrease in darkness in the same order and vice versa, provided that the relative increase or decrease in each case is identical.
**Corollary D2.5:** A reversal of the order of elements of a two element set of real numbers changes its shape unless the elements are identical and the shape is zero.

## 1.3 Extension to Larger Sets

This definition of shape for a two element set of real numbers could suffice for larger sets because an ordered set of N elements can be decomposed uniquely into ${}^{N}C_{2}$ ordered subsets of two elements. A change in the values or the order of the set will involve a change in one or more of these two-element subsets. However, such an extended definition of shape for larger sets will be inadequate because it does not capture "global" perception of shape that includes phenomena such as foveation or focussing. It only captures shape as a union of local shape information each of that assumes the same reference level. The measure of shape in equation 1 is adaptive to the local brightness because it is a ratio, however, the zero of shape is still the same for every subset. In practice, a large ordered set is often perceived "globally" over an extended portion of more than two elements. Such perception is usually a weighted average with reduced contributions from elements that are further away from a point called the focus or foveation point. Because the shape of an arbitrary ordered set is defined as the union of the shapes of all its possible ordered subsets, the focus point in any subset can be chosen, without loss of generality, as the centre or that point about which there are an equal number of elements both below and above in the given order of elements. For a subset with odd number of elements, this point will correspond to one of the elements. For a subset with an even number of elements the focus lies between two elements.
The extended definition of shape for a larger set should measure the "change" when the order of elements is reversed about the foveation point. It will then be zero for symmetric sets - not just sets of constants. This does not imply that different symmetric sets cannot be perceived as different in shape - they will have different shapes in other subsets. The extended definition of shape is thus a measure of relative asymmetry. The shapes of all ordered sets can be characterised by measuring relative asymmetry of all possible ordered subsets. For a two-element set the asymmetry is measured as a difference of the values. For a larger set, the asymmetry can be measured as a difference between the weighted sums of values on either side of the foveation point or centre. The optimal choice of weights depends on a number of criteria discussed below.

# 2. ORDER SENSITIVITY THEOREMS

**Theorem T1:** If $\{w(i), i = 1,2,...., N\}$ are weights and $\{x(i), i = 1,2,...., N\}$ are elements of an ordered set, the weighted sum $\sum_{i=1}^{N} w(i)x(i)$ is different for every pair-wise exchange of $x(m)$ and $x(n)$ where $x(m) \neq x(n)$ provided every pair of weights $w(m)$ and $w(n)$ is unique.
**Proof:** Consider the reversal of elements with indices *m* and *n*. The weighted sum will not change if:

$$w(m)x(m) + w(n)x(n) \neq w(m)x(n) + w(n)x(m) \quad \text{or}$$

$$w(m)[x(m) - x(n)] - w(n)[x(m) - x(n)] \neq 0 \quad \text{or}$$

$$[w(m) - w(n)][x(m) - x(n)] \neq 0 \quad \text{or}$$

$$w(m) \neq w(n) \quad \text{given} \quad x(m) \neq x(n) .$$

Any permutation of elements of an ordered set can be considered as a series of pair-wise reversals of elements. If uniqueness to a combination of two pair-wise reversals is desired, then the differences of weights $[w(m) - w(n)]$ must also be unique for every possible *m* and *n*. Carrying the argument forward to permutations involving three reversals and so on, the following theorem is proved.
**Theorem T2:** If $\{w(i), i = 1,2,...., N\}$ are weights and $\{x(i), i = 1,2,...., N\}$ are elements of an ordered set, the weighted sum $\sum_{i=1}^{N} w(i)x(i)$ is unique for every unique permutation provided every pair of weights $w(m)$ and $w(n)$ is unique and their differences up to order N-1 are unique.
**Corollary T1.1:** A weighted sum of an ordered set of elements using a monotonically increasing or monotonically decreasing set of weights is unique to the reversal of order of pairs of elements in the set.
**Definition D3:** An ordered set can be represented by indices with the foveation point having index 0, and other elements having index equal to the integer value of its position to the right (positive) or left (negative) of the foveation point. Such a set will be referred to as a "centred" set.
**Corollary T1.2:** A weighted combination of the elements of a centred set with the weights equal to the corresponding index is unique to the reversal of order of pairs of elements. Such a weighted sum is in fact the first moment.
**Corollary T2.1:** A weighted combination of the elements of a set indexed from 0 to N-1 with the weights proportional to the N-th power of the corresponding index is unique to any permutation of the N-element ordered set. Such a weighted sum is in fact the N-th order moment.
**Corollary T2.2:** A weighted combination of the elements of a *centred* set indexed with the centre as 0 and with the weights proportional to the N-th power of the corresponding index is *not* unique to any permutation of the N-element ordered set. Such a weighted sum is in fact the N-th order central moment. This is owing to the non-uniqueness of $i^{2M}$, which assumes the same value for $\pm i$ .

Thus, central moments of even order lose uniqueness to mirror image inputs. Central moments of odd order are zero-valued for inputs that are symmetric about the centre. A central moment of any given order is therefore not unique to all permutations of a given input.

Which other weight sequences satisfy the condition of sensitivity to permutations of the input? Do they have to be monotonically increasing or decreasing? Are they sensitive to any change in the input?

It can be shown that weights that are inversely proportional to the index also satisfy a limited order sensitivity property. Such a weighted sum, in fact, corresponds to computation of a Hilbert transform coefficient in a discrete domain. For a continuous-time function, the Hilbert transform operation is a convolution with the function $1/t$ (except for a scalar constant $1/\pi$). All derivatives of $1/t$ exist except at $t$ = 0. At any given $t = t_0$, the Hilbert transform coefficient is an integral that measures asymmetry about this point and retains sensitivity to the order of values in the input. The Hilbert transform suffers from the same problem as central moments because even order differences (or derivatives) are not unique.

Weights that decay with increasing distance from the foveation point are of particular interest because they yield a bounded measure for bounded sets (stability). They are also consistent with the phenomenon of decreasing attention to detail or loss of sensitivity away from the point of focus.

From theorem T1 it can be seen that even for a two-element set, any weighted combination of the elements could have been used as a measure of shape, if sensitivity to the order of elements was the only desirable criterion. It must be noted, however, that the definition in D1 satisfies additional desirable criteria such as the shape of a constant set being zero even when the constant itself is not zero and conformity to Weber's law of perception. The definition in D1 can be considered a weighted sum given by

$$\psi_2 = w(1)x(1) + w(2)x(2) \qquad (2)$$

where

$$w(1) = \frac{1}{x(1)+x(2)} \quad \text{and} \quad w(2) = \frac{-1}{x(1)+x(2)}.$$

The weights satisfy the conditions

$$w(1) + w(2) = 0 \qquad \text{and} \quad w(i) \propto \frac{1}{\frac{1}{2}\sum_{i=1}^{2} x(i)} \qquad (3)$$

The second constraint implies that the weights are inversely proportional to the average brightness of elements in the set. The average brightness is itself not a measure of shape according to the above development because its weighting of the elements is constant. At this point, a name can be given to such a measure - it will be referred to as a "statistical measure" because it is completely independent of the order of the elements.

## 3. STATISTICAL AND SHAPE MEASURES

**Definition D4:** A statistical measure of an ordered set of real numbers is a scalar function or property of the set that is completely independent of the order of elements in the set.

For example, mean, median and mode are all values that describe a set of numbers but they are independent of the order. The histogram that describes the frequency of occurrence of values in the elements of the set is another example of a statistical measure. The histogram is a vector from which several scalar statistical measures can be derived.

**Definition D5:** A statistical-shape "hybrid" measure of an ordered set of real numbers is a scalar function or property which is neither unique to the order of the elements in the set nor completely independent of the order, that is, neither a shape measure nor a statistical measure.

The first order moment of a set of elements is not, in general, a complete shape measure because the differences of weights are not unique.

**Definition D6:** A complete shape measure of an ordered set of real numbers is a scalar function or property which is unique to the order of the elements in the set.

From the above discussion, it follows that the N-th order moment is a complete shape measure of an ordered set. N-th order central moments of centred sets are not complete shape measures, neither is the Hilbert transform coefficient.

In general, statistical measures can provide robustness at the expense of uniqueness, while shape measures provide uniqueness to the order of elements.

The constraints on the weights of a shape measure for a two-element set (equations 3) can be generalized for an extended definition to larger sets. Then for a set of N = 2M (or 2M+1) elements,

$$\sum_{i=-M}^{M} w(i) = 0 \qquad (4)$$

and

$$w(i) \propto \frac{1}{\frac{1}{N}\sum_{k=-M}^{M} x(k)} \qquad (5)$$

If the point of foveation is an element of the set (odd number of elements) w(0) is chosen as 0 and the corresponding element will not contribute to shape.

The second constraint is satisfied easily by dividing all weights by the average brightness. There are several possible choices that satisfy the first constraint. If the indices are $\{i = -M, -M+1, ...., -1, 1, ...., M-1, M\}$ then the possible choices include

$$w(i) = \frac{i^M}{\sum_{k=-M}^{M} x(k)} \quad \text{or} \quad \frac{1/i}{\sum_{k=-M}^{M} x(k)} \quad \text{or} \quad \mathrm{sgn(i)}\frac{\exp(-|i|)}{\sum_{k=-M}^{M} x(k)} \qquad (6)$$

By power series expansion such monotonically decreasing functions can be represented in terms involving powers of $1/i$. All these functions are measures of asymmetry about the centre. Any of them are acceptable as a measure of shape in the subset. Adopting the second choice, of weights inversely proportional to the order index from the centre,

**Definition D7:** For an N-element set of real numbers $\{x_{-M}, x_{-M+1}, ...., x_{-1}, x_0, x_1, ...., x_{M-1}, x_M\}$ or $\{x_{-M}, x_{-M+1}, ...., x_{-1}, x_1, ...., x_{M-1}, x_M\}$ the partial shape it contributes as a subset is measured by:

$$\psi_N = \frac{\sum_{i=-M}^{M} \frac{1}{i} x(i)}{\sum_{k=-M}^{M} x(k)} \tag{7}$$

# 4. FEATURES AND CLASSIFICATION

Definitions D1 and D7 can be used to represent the shape of an arbitrary one-dimensional ordered set using a set of scalar shape measures, one for each ordered subset. The complete computation of this set of measures has exponential complexity because there are $2^N$ such subsets. It will not in general be of much use in classification or compression; in fact, it expands a set of N real numbers into a larger set of real numbers. Procedures and criteria to prune and combine these measures into lower dimensional representations are required to introduce efficiency as well as robustness. Ideally, a scalar whose value is unique to every different shape and has a high variance between desirable classes of shapes and a low variance between acceptable degradations of a given shape is the preferred representation or "feature". Often pattern recognition methods have to be satisfied with vectors of features that are then used by a classifier for discriminating the classes of shapes. Classification utilizes shape and statistical information in the "feature" space.

## 4.1 Multi-dimensional Signals

For two-dimensional images or multi-dimensional signals, the above discussion and definitions hold with the order extended from one-dimension to multiple dimensions and signals being viewed as sets with multiple indices. Since the definition of shape involves contribution from all possible subsets, theoretically, finite extent multi-dimensional sets can be treated as one-dimensional sets after collapsing the dimensions into one in a prescribed manner. For example, an image can be collapsed into a vector by concatenating consecutive rows.

**EXAMPLE:**

We illustrate how the theory may be applied by a 2D example. Consider the four bi-level patterns shown below.

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | | 0 | 1 | 0 | 1 | | 0 | 1 | 1 | 1 | | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | | 0 | 0 | 1 | 0 | | 0 | 0 | 0 | 1 | | 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 0 | | 0 | 1 | 0 | 1 | | 0 | 0 | 0 | 1 | | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 |

(a) (b) (c) (d)

**Figure 1. Bi-level patterns.**

In order to apply the theory to multidimensional signals we must convert them to one-dimensional ordered sets according to some prescribed manner. Let us choose this as concatenation of rows starting from the top left corner, yielding vectors x(n) from inputs. We will examine the "shape" content and discriminability of these patterns limited to the second order description given by ordered pairs. Ordering these two-element subsets [x(k) x(l)] with k less than or equal to l, there are 128 subsets from each. Of these, exactly 55 have non-zero shape for each pattern.

| | Shape (a) | Shape (b) | Shape (c) | Shape (d) |
|---|---|---|---|---|
| $\psi_2 = 1$ | 35 | 20 | 14 | 26 |
| $\psi_2 = -1$ | 20 | 35 | 41 | 29 |

As seen from the table, in terms of such a measure, shape (a) is furthest from its complement shape (b), and shape (b) lies between shapes (c) and (d). It must be noted that the description above is only second order. Since our system is based on asymmetry, transposing the matrices has the same effect as complementing – it changes the symmetry in the second order subsets.

# 5. CONCLUSION

In this paper, a new approach is presented to define shape and measures of shape. An attempt is made to place one-dimensional and multi-dimensional pattern recognition as well as different approaches to extracting features dependent on shape and robust to noise and transformations on the same unified analytical framework. Further work in this direction will enable meaningful comparisons of different pattern recognition techniques in terms of discriminability and robustness.

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