ON ESTIMATING RANDOM AMPLITUDE CHIRP SIGNALS

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ABSTRACT

This paper considers the problem of estimating the parameters of chirp signals with randomly time-varying amplitude. Two methods for solving this problem are presented. First, a nonlinear least-squares approach (NLS) is proposed. It is shown that by minimizing the NLS criterion with respect to all samples of the time-varying amplitude, the problem reduces to a two-dimensional maximization problem. A theoretical analysis of the NLS estimator is presented and an expression for its asymptotic variance is derived. It is shown that the NLS estimator has a variance very close to the Cramér-Rao Bound. The second approach combines the principles behind the High-Order Ambiguity Function (HAF) and the NLS approach. It provides a computationally simpler but suboptimum estimator. A statistical analysis of this estimator is also carried out. Numerical examples attest to the validity of the theoretical analysis and establish a comparison between the two proposed methods.

1. INTRODUCTION AND OUTLINE

Retrieving the parameters of chirp signals embedded in multiplicative and additive noise is a topic of considerable interest in many practical situations. Consider the radar application where a target is illuminated. Then, the transmitted signal will be affected by two different phenomena. First, due to the distance and relative motion between the target and the receiver, the phase of the signal will be shifted. This phase shift can be adequately modeled as $\phi(t) = a_0 + a_1 t + a_2 t^2$ provided that the motion is continuous and differentiable. The parameters a_1 and a_2 are either related to speed and acceleration, or range and speed, depending on what the radar is intended for and on the kind of waveforms transmitted [1]. Additionally, the signal will experience amplitude distortion caused either by target fluctuation or scattering of the medium (e.g., fading). This manifests itself as a random time-varying amplitude $\alpha(t)$ which can be viewed as an unwanted phenomenom (hence the terminology multiplicative noise often used in the literature). The cases of *constant amplitude chirp signals* and *exponential signals with time-varying amplitude* have been addressed thoroughly (see [2, 3] and references therein). In contrast, the literature is more scarse regarding chirp signals with random, time-varying amplitudes. The case of deterministic amplitude $\alpha(t)$ is treated in [4]-[5] whereas [6]-[7] consider random amplitudes. Cramér-Rao Bounds are derived in [6] when $\alpha(t)$ is a stationary process whose covariance matrix depends on a finite dimensional parameter vector. A broader class of random amplitudes is considered in [7] where cyclostationary solutions are investigated to solve the problem. More precisely, for a chirp signal, use of the cyclic 2nd-order moment is advocated.

Two approaches are presented in this paper. The first relies on Nonlinear Least-Squares estimation of the chirp parameters. Since this approach may be computationally intensive for certain applications, a simpler approach is proposed which borrows ideas from the HAF and the NLS estimator.

2. NLS ESTIMATION

The signal to be dealt with herein is given by

$$\begin{aligned} y(t) &= \alpha(t) e^{i\left(a_0 + a_1 t + a_2 t^2\right)} + n(t) \quad t = 0, \cdots, N-1 \\ &= s(t) + n(t) \end{aligned}$$

where $\alpha(t)$ is assumed to be a real-valued Gaussian stationary process and $n(t)$ is a white complex circular Gaussian process with zero mean and variance σ^2, i.e., $\mathsf{E}\{n(t)n(t+\tau)\} \equiv 0$, $\mathsf{E}\{n^*(t)n(t+\tau)\} = \sigma^2\delta(\tau)$. Additionally, $n(t)$ is assumed to be independent of $\alpha(t)$. The NLS approach consists of estimating the parameters a_0, a_1, a_2, as well as all samples $\{\alpha(t)\}_{t=0,\cdots,N-1}$ of the time-varying amplitude by minimizing the following criterion :

$$J(\boldsymbol{\alpha}, \mathbf{a}) = \frac{1}{N} \sum_{t=0}^{N-1} \left| y(t) - \alpha(t) e^{i\left(a_0 + a_1 t + a_2 t^2\right)} \right|^2 \qquad (1)$$

where $\boldsymbol{\alpha} = [\alpha(0), \cdots, \alpha(N-1)]^T$ and $\mathbf{a} = [a_0, a_1, a_2]^T$. In the next proposition, we show how estimates of $\boldsymbol{\alpha}$ and $\mathbf{a}$ can be obtained.

Proposition 1. *The vectors* $\boldsymbol{\alpha}$ *and* $\mathbf{a}$ *which minimize (1) are given by*

$$\widehat{a}_1, \widehat{a}_2 = \arg\max_{a_1,a_2} \frac{1}{N}\left|\sum_{t=0}^{N-1} y^2(t) \times e^{-i2(a_1 t + a_2 t^2)}\right|^2 \quad (2)$$

$$\widehat{a}_0 = \frac{1}{2}\text{angle}\left\{\sum_{t=0}^{N-1} y^2(t) \times e^{-i2(\widehat{a}_1 t + \widehat{a}_2 t^2)}\right\} \quad (3)$$

$$\widehat{\alpha}(t) = \text{Re}\left\{y(t) \times e^{-i(\widehat{a}_0 + \widehat{a}_1 t + \widehat{a}_2 t^2)}\right\} \quad (4)$$

Proof : see [8].

Note that a "true" NLS estimator would proceed by minimizing (1) with respect to $\mathbf{a}$ and the parameter vector $\boldsymbol{\lambda}$ on which $\alpha(t)$ would depend. The approach we propose tacitly considers that the realization of $\alpha(t)$ is frozen and has to be estimated. This in turn decreases the computational complexity since only a 2D maximization problem needs to be solved when minimizing with respect to $\boldsymbol{\alpha}$ and not wrt $\boldsymbol{\lambda}$. Additionally, it should be emphasized that the present approach does not rely on any assumed structure for the amplitude; hence, it has the desirable property of being applicable to a wide class of signals.

Remark 1. In the constant amplitude case (i.e., $\alpha(t) \equiv \alpha_0$), the estimate of α_0 would be an average, e.g., $\widehat{\alpha}_0 = \frac{1}{N}\left|\sum_{t=0}^{N-1} y(t) \times e^{-i(\widehat{a}_1 t + \widehat{a}_2 t^2)}\right|$. In the time-varying scenario, each sample of $\alpha(t)$ is estimated (see (4)) which leads to the squaring of the data.

Remark 2. The estimates of $\mathbf{a}$ as given by (2) and (3) are equivalent to those that would have been obtained by solving the following minimization problem

$$\left\{\widehat{\mathbf{a}}, \widehat{A}\right\} = \arg\min_{A,\mathbf{a}} \sum_{t=0}^{N-1} \left|y^2(t) - Ae^{i2\left(a_0 + a_1 t + a_2 t^2\right)}\right|^2 \quad (5)$$

Moreover, it can be readily verified that the estimate $\widehat{A}$ of the "signal amplitude" as obtained from (5) is a consistent estimate of $r_\alpha(0) = \mathsf{E}\left\{\alpha^2(t)\right\}$. Hence, the NLS estimator "views" the signal as

$$y^2(t) = r_\alpha(0) e^{i2(a_0 + a_1 t + a_2 t^2)} + \Delta(t) \quad (6)$$

We now analyze the performance of the estimates of a_1 and a_2 as given by (2) and derive an expression for their variances.

Proposition 2. *The asymptotic variances of* $\widehat{a}_1$ *and* $\widehat{a}_2$ *in (2) are given by*

$$\mathsf{var}\left(\widehat{a}_1\right) \simeq \frac{96}{N^3}\frac{1}{SNR}\left[1 + \frac{1}{2}SNR^{-1}\right]$$
$$\mathsf{var}\left(\widehat{a}_2\right) \simeq \frac{90}{N^5}\frac{1}{SNR}\left[1 + \frac{1}{2}SNR^{-1}\right] \quad (7)$$

where $SNR = r_\alpha(0)/\sigma^2$.

Proof : see [8].

It should be stressed that although $\alpha(t)$ may be colored, the variance expression (7) involves only the zero-lag term $r_\alpha(0)$. Additonally, similar to the constant amplitude case, the variances of $\widehat{a}_1$ and $\widehat{a}_2$ are seen to be of orders $1/N^3$ and $1/N^5$, respectively. Finally, it is of interest to compare the above expressions with the Cramér-Rao Bounds (CRB) derived in [6]. In the high SNR case, the CRB's are given by (see [6, eq. (89)])

$$\mathsf{CRB}\left(\widehat{a}_1\right) \simeq \tfrac{96}{N^3}\tfrac{1}{SNR} \quad \mathsf{CRB}\left(\widehat{a}_2\right) \simeq \tfrac{90}{N^5}\tfrac{1}{SNR} \quad (8)$$

Comparing (7) with (8), it is seen that the NLS estimator provides nearly efficient estimates.

3. HAF-BASED APPROACH

Since the NLS estimator involves a 2D maximization, we consider a simpler approach. Specifically, we would like to apply the original HAF based approach of Peleg and Porat. Consider first the noiseless case. It is readily verified that

$$s^*(t)s(t+\tau) = \alpha(t)\alpha(t+\tau)e^{ia_1\tau}e^{ia_2\tau^2}e^{i2a_2 t\tau} \quad (9)$$

where τ is some positive integer ($\tau > 0$). Hence, $s_2(t;\tau) = s^*(t)s(t+\tau)$ is an exponential signal with time-varying amplitude $\beta(t;\tau) = \alpha(t)\alpha(t+\tau)$. In the noisy case, we obtain

$$\begin{aligned} y_2(t;\tau) &= y^*(t)y(t+\tau) \\ &= s^*(t)s(t+\tau) + n_2(t;\tau) \end{aligned} \quad (10)$$

where $n_2(t;\tau) = s^*(t)n(t+\tau) + n^*(t)s(t+\tau) + n^*(t)n(t+\tau)$ is a zero-mean (since $\tau > 0$) process with covariance

$$\mathsf{E}\left\{n_2^*(t;\tau)\, n_2(t+r;\tau)\right\} = \left[2\sigma^2 r_\alpha(0) + \sigma^4\right]\delta(r) \quad (11)$$

Therefore, $y_2(t;\tau)$ is an exponential signal with random time-varying amplitude $\beta(t;\tau) = \alpha(t)\alpha(t+\tau)$ in complex zero-mean white noise $n_2(t;\tau)$. However, the distributions of $\beta(t;\tau)$ and $n_2(t;\tau)$ are quite complicated to obtain; hence an optimal (e.g., Maximum Likelihood) approach is to be forgotten. Thus, we are naturally led to using a NLS approach with $y_2(t;\tau)$ as the data. The steps involved in the estimation of a_1 and a_2 are now described.

Step 1 For a given τ, compute $y_2(t;\tau) = y^*(t)y(t+\tau)$. Then estimate a_2 as

$$\begin{aligned} \widehat{a}_2 &= \frac{1}{2\tau}\arg\min_{\boldsymbol{\beta},\varphi,\omega} \frac{1}{N}\sum_{t=0}^{N-1}\left|y_2(t;\tau) - \beta(t;\tau)\, e^{i(\omega t + \varphi)}\right|^2 \\ &= \frac{1}{2\tau}\arg\max_{\omega} \frac{1}{N}\left|\sum_{t=0}^{N-1} y_2^2(t;\tau)\, e^{-i2\omega t}\right| \end{aligned} \quad (12)$$

which follows from Proposition 1. Note that $\widehat{a}_2$ can be obtained via the Fast Fourier Transform of $y_2^2(t;\tau)$.

Step 2 Once $\widehat{a}_2$ is available demodulate $y(t)$ to obtain

$$\begin{aligned} z(t) &= y(t) \times e^{-i\widehat{a}_2 t^2} \\ &\simeq \alpha(t)\, e^{i(a_0+a_1 t)} + \widetilde{n}(t) \end{aligned} \quad (13)$$

where $\widetilde{n}(t)$ combines the estimation errors in $\widehat{a}_2$ and the effect of additive noise. a_1 is thus obtained as

$$\begin{aligned} \widehat{a}_1 &= \arg\min_{\boldsymbol{\alpha},\varphi,\omega} \frac{1}{N}\sum_{t=0}^{N-1} \left| z(t) - \alpha(t)\, e^{i(\omega t+\varphi)} \right|^2 \\ &= \arg\max_{\omega} \frac{1}{N} \left| \sum_{t=0}^{N-1} z^2(t)\, e^{-i2\omega t} \right| \end{aligned} \quad (14)$$

We note that the previous approach is simpler than the NLS approach as it only involves lag products and FFT's. In the next section, we will examine the trade-offs between statistical accuracy of the NLS estimator and computational simplicity of the HAF estimator.

Remark 3. It can be readily verified that the estimate $\widehat{a}_2$ in (12) implicitly relies on a 4th-order transformation of the data since $\widehat{a}_2 = \frac{1}{2\tau}\arg\max_\omega \frac{1}{N}\left|\sum_{t=0}^{N-1}\left[y^*(t)\, y(t+\tau)\, e^{-i\omega t}\right]^2\right|$. In fact, the HAF-based scheme amounts to using the "classical" HAF-estimator (i.e., the estimator originally designed for constant amplitude chirps) but with $y(t)$ replaced by $y^2(t)$.

Remark 4. A statistical analysis of the HAF-based estimate $\widehat{a}_2$ in (12) is carried out in [8]. Assuming that $N - \tau \gg 1$, the large sample variance of the HAF estimate of a_2 is given by

$$\mathsf{var}(\widehat{a}_2) \simeq \frac{6}{(N-\tau)^3}\frac{D(\tau)}{8\tau^2 m_{4,\alpha}^2(0,\tau,\tau)} \quad (15)$$

with

$$\begin{aligned} D(\tau) &= 4\sigma^2 m_{6,\alpha}(0,0,0,\tau,\tau) + 2\sigma^4 m_{4,\alpha}(0,0,0) \\ &\quad + 8\sigma^4 m_{4,\alpha}(0,\tau,\tau) + 8\sigma^6 r_\alpha(0) + 2\sigma^8 \\ &\quad - \left[4\sigma^2 m_{6,\alpha}(0,\tau,\tau,2\tau,2\tau) + 2\sigma^4 m_{4,\alpha}(0,2\tau,2\tau)\right] \\ &\quad \times \frac{(N-2\tau)(N^2-4\tau N+\tau^2)}{(N-\tau)^3}\mathbf{1}(N-2\tau) \end{aligned}$$

where $\mathbf{1}(.)$ is the unit step function. Observe that the variance of the HAF-based estimator depends on τ and the fourth and sixth-order moments of $\alpha(t)$. Hence, derivation of an optimal τ solely as a function of N, like in the constant amplitude case, appears not to be directly feasible. However, the form of (15) suggests than an optimal τ should be close to $0.5N$.

4. NUMERICAL EXAMPLES AND CONCLUSIONS

In this section, Monte-Carlo simulations are performed to illustrate the performances of the two proposed estimators. In all the simulations, $\alpha(t)$ is generated as a zero-mean $AR(2)$ process with poles at $\rho e^{\pm i2\pi f}$ and the additive noise is white Gaussian with variance σ^2. The AR pole parameters are $\rho = 0.95$ and $f = 0.01$, unless otherwise stated. The Signal to Noise Ratio is defined as $SNR = r_\alpha(0)/\sigma^2$. The chirp parameter vector is $\mathbf{a} = 2\pi\left[0.1 \quad 0.18 \quad 3\times10^{-4}\right]^T$. Figure 1 displays the influence of τ onto the performance of the HAF estimator. It can be observed that the variance first decreases, then reaches a constant floor (for τ between $0.2N$ and $0.5N$) before increasing when τ is large. Also, note that, for the optimal choice of τ the variance of the HAF estimate is about $3.6 \times \mathsf{CRB}$.

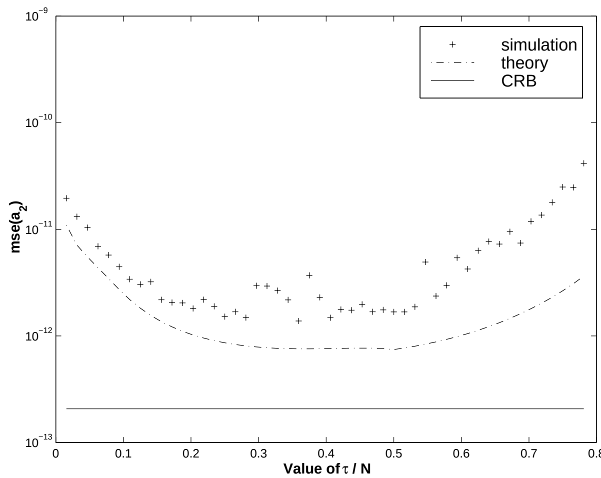

Figure 1: Theoretical and empirical variances of $\widehat{a}_2$ in (12) versus τ. $N = 256$ and $SNR = 10$dB.

Figures 2 to 5 display the influences of N, SNR, ρ and f respectively on the performance of the estimators. In these figures, τ is chosen as $0.4N$. Since the theoretical variance of the NLS estimator are very close to the CRB's, only these latter are plotted. The following points are worth being noted :

- As predicted by the theoretical analysis, the NLS estimator is seen to come close to the Cramér-Rao Bound, provided that N and SNR are sufficiently large (typically $N \geq 256$ and $SNR \geq 10dB$). The HAF estimate performs as well as the NLS for small N or low SNR. In contrast, the NLS performs better for large N or high SNR.

- The HAF estimator (and in certain respect the NLS estimator) exhibits the threshold effect in SNR which is inherent to nonlinear transformations and has already been reported in other studies on the same kind of algorithms.

- The performance remains stable even if the bandwidth of $\alpha(t)$ increases, e.g., as ρ decreases or f increases.

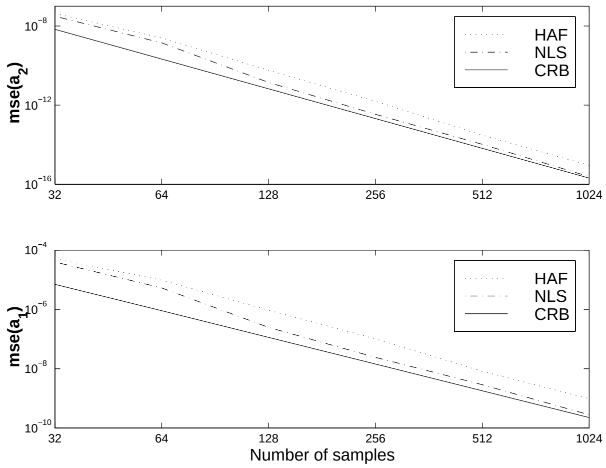

Figure 2: CRB (solid lines) and Mean Square Errors of $\widehat{a}_1$ and $\widehat{a}_2$ versus number of samples. $SNR = 10$dB.

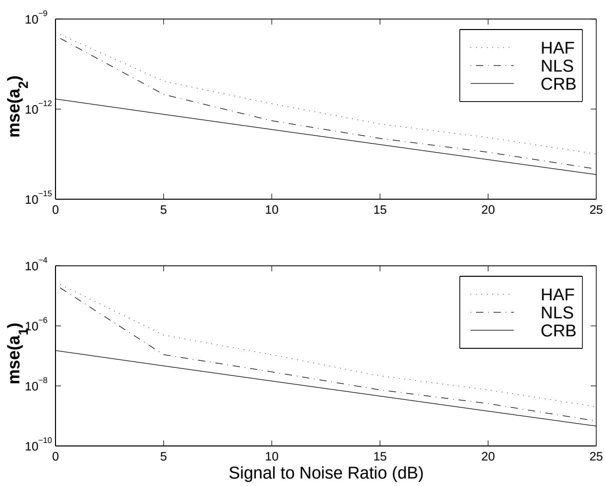

Figure 3: CRB (solid lines) and Mean Square errors of $\widehat{a}_1$ and $\widehat{a}_2$ versus SNR. $N = 256$.

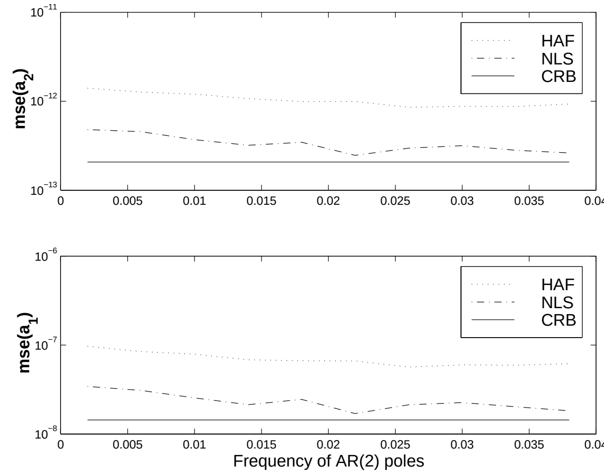

Figure 4: CRB (solid lines) and Mean Square errors of $\widehat{a}_1$ and $\widehat{a}_2$ versus f. $N = 256$ and $SNR = 10$dB.

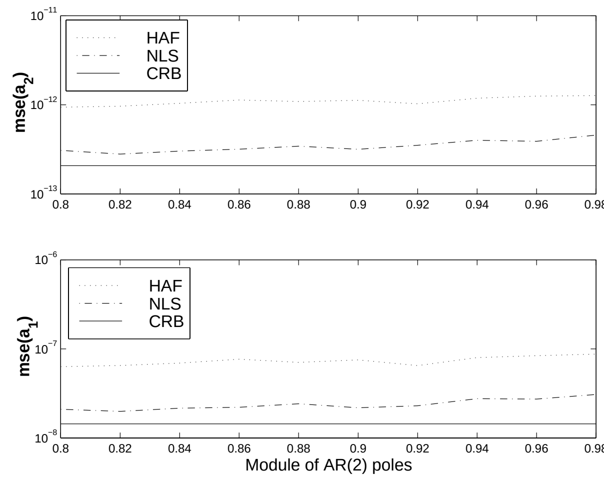

Figure 5: CRB (solid lines) and Mean Square errors of $\widehat{a}_1$ and $\widehat{a}_2$ versus ρ. $N = 256$ and $SNR = 10$dB.

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