OPTIMAL DMT TRANSCEIVERS OVER FADING CHANNELS

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ABSTRACT

Recently DFT based discrete multitone modulation (DMT) systems have been widely applied to various applications. In this paper we study a broader class of DMT systems using more general unitary matrices instead of DFT matrices. For this class we will show how to design the optimal DMT systems over fading channels with colored noise. Examples will be given to show the improvement over the traditional DFT based DMT system. In addition we introduce a modified DFT based DMT system. The new system has the same complexity but better noise rejection property.

1. INTRODUCTION

Recently there has been considerable interest in applying the discrete multitone modulation (DMT) technique to high speed data transmission over fading channels such as ADSL and HDSL [1][2]. In the widely used DFT based DMT system (Fig. 1), the channel is divided into a number of subchannels by using DFT matrices. High speed data transmission can be obtained at a relatively low cost [1]. In the DFT based DMT system, a certain degree of redundancy known as cyclic prefix is added to achieve ISI free transmission over fading channels [1][2].

In [6] Kasturia et. al advance the DMT system using more general unitary matrices instead of DFT matrices. When the channel noise is AWGN, the authors show that the optimal transmitter and receiver are composed of eigenvectors of some Toeplitz matrices associated with the channel impulse response. However for applications such as ADSL, the dominating noise source is usually crosstalk and the noise is colored [1].

In this paper, we will use a polyphase approach [2] to study the DMT system. Using this approach, we will derive a modified DFT based DMT system which has a better noise rejection property than the traditional DFT based system at the same cost. Moreover optimal transceiver for colored noise will be studied in details. In particular, we will show how to assign bits among the channel so that the total tranmitting power can be minimized for a given bit rate. Based on the optimal bit allocation the design of the optimal transceiver is derived. Futhermore we will see that although the DFT based DMT system is not optimal, for AWGN fading channels, its asymptotical performance approaches that of the optimal system when the number of channels is large.

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2. POLYPHASE REPRESENTATION OF DMT SYSTEMS

Consider Fig. 2, where an M-channel DMT system is shown. Usually the channel is modelled as an LTI filter $C(z)$ with additive noise $e(n)$. Assume that $C(z)$ is an FIR filter of order L (a reasonable assumption after channel equalization) and $e(n)$ is a zero-mean WSS random process. For a given channel number M, the interpolation ratio N is chosen as $N = M + L$. As redundancy is introduced in this case, we say the system is over interpolated. The filters $F_k(z)$ and $H_k(z)$ are called transmitting and receiving filters respectively. In the DMT system, $F_k(z)$ and $H_k(z)$ have length $\leq$ the interpolation ratio N.

Using polyphase decomposition the DMT system can be redrawn as in Fig. 3 [2]. The tranmitter $\mathbf{G}$ is an $N \times M$ contant matrix; the kth column of $\mathbf{G}$ contains the coefficients of the transmitting filter $F_k(z)$. The receiver $\mathbf{S}$ is an $M \times N$ contant matrix; the kth row of $\mathbf{S}$ contains the coefficients of the receiving filter $H_k(z)$. The matrix $\mathbf{C}(z)$ is an $N \times N$ pseudo circulant matrix [4] with the first column given by

$$\left(c_0 c_1 \cdots c_L 0 \cdots 0\right)^T$$

where c_n is the channel impulse response.

Perfect reconstruction condition. From Fig. 3, we see that the overall transfer function of the DMT system is

$$\mathbf{T}(z) = \mathbf{S}\mathbf{C}(z)\mathbf{G}. \tag{1}$$

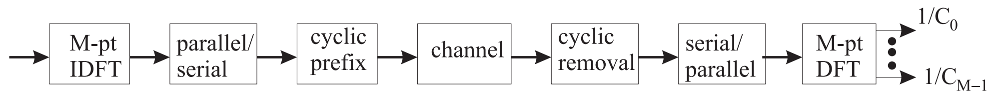

Figure 1: An M-channel DFT based DMT system

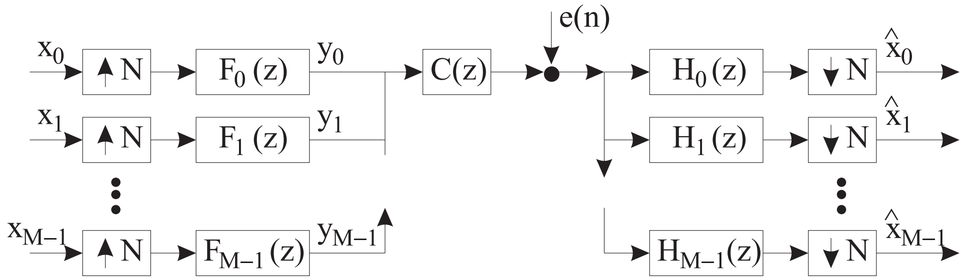

Figure 2: An M-channel DMT system over a fading channel.

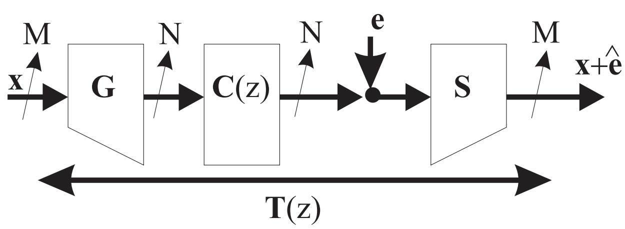

Figure 3: The polyphase representation of the DMT system

When $\mathbf{T}(z) = \mathbf{I}$ we say the DMT system has ISI free property or perfect reconstruction (PR); the outputs are identical to the inputs except delays in the absence of channel noise.

Interchange of the transmitting and receiving filters

One immediate advantage of the polyphase approach is that it tells us how to interchange the trasmitting and receiving filters and still preserves the PR property. To see this, observe that the pseudo-circulant matrix $\mathbf{C}(z)$ satisfies,

$$\mathbf{C}(z) = \mathbf{J}_N \mathbf{C}^T(z) \mathbf{J}_N, \tag{2}$$

where $\mathbf{J}_N$ is the $N \times N$ reversal matrix. Using (1) and (2), we can exchange the transmitter and receiver: if a DMT system with transmitter receiver pair ($\mathbf{G}$, $\mathbf{S}$) is perfect, then the DMT system ($\mathbf{S}^T$, $\mathbf{G}^T$) is also perfect. This implies that, we can exchange the transmitting filters and receiving filters and the system is still perfect even when the channel is a fading channel.

3. MODIFIED DFT BASED DMT SYSTEMS

The block diagram of DFT based DMT system is shown in Fig. 1. The transmitter performs two operations: computing the M-point inverse DFT of each input block and adding L cyclic prefix. The redundancy allows the receiver to remove ISI and the overall system is perfect. The receiver consists of an M-point DFT matrix and M scalars $1/C_k$, for $k = 0, 1, \cdots, M-1$, where C_k are the M-point DFT of the channel impulse response. It has the great advantage that the whole system is almost channel independent except the M scalars $1/C_k$.

One can verify that the transmitter $\mathbf{G} = [\mathbf{W}\mathbf{W}_1]^\dagger$, where $\mathbf{W}$ is the $M \times M$ DFT matrix with $[\mathbf{W}]_{mn} = W^{mn}$ and $W = e^{-j2\pi/M}$ and $\mathbf{W}_1$ is a submatrix of $\mathbf{W}$ that contains the first L columns of $\mathbf{W}$. The receiver is $\mathbf{S} = \mathbf{\Lambda}^{-1}[\mathbf{0}\ \mathbf{W}]$, where $\mathbf{\Lambda}$ is the diagonal matrix $diag(C_0, C_1, \cdots, C_{M-1})$. Note that the kth row of $\mathbf{S}$ contains the coefficients of the kth receiving filter $H_k(z)$. So the receiving filters are DFT filters of length M and hence the frequency responses will have a main lobe of width $2\pi/M$.

Now if we exchange the transmitter and the receiver (with slight modification), we get the modified DFT based DMT system.

$$\mathbf{G} = \begin{pmatrix} \mathbf{0} \\ \mathbf{W}^\dagger \end{pmatrix}, \mathbf{S} = \mathbf{\Lambda}^{-1}[\mathbf{W}\mathbf{W}_1].$$

The modified system has the same complexity as the conventional case. But the new receiving filters are DFT filters with length N instead of M in the conventional case. The allows the new system to enjoy additional advantages. First, the new receiving filter have narrower bandwidth $2\pi/N$. Fig. 4 gives a comparison of the conventional and new receiving filters for the same transmitting power. Only the first receiving filters of these two systems are shown as the other receiving filters are shifted versions of the first filter. The narrower main lobe in the modified case gives better performance in rejecting out of band noise. Moreover, the new receiving filters have longer length. The channel noise will be averaged over a longer block and the effect of impulsive noise will be reduced.

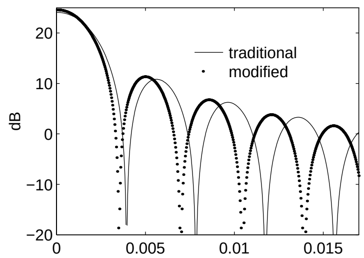

Figure 4: The magnitude responses of the first receiving filters in the conventional DFT based DMT system and the modified system for $L = 32$ and $M = 256$. (Frequency normalized by 2π)

4. GENERALIZED PERFECT DMT SYSTEMS

The transmitter of the modified DFT system can be viewed the coding of the input block using DFT vectors plus padding of L zeros. We can generalize the system by letting the transmitter be

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_0 \\ \mathbf{0} \end{pmatrix} \tag{3}$$

where $\mathbf{G}_0$ is an arbitary $M \times M$ unitary matrix. With $N = M + L$, we can partition $\mathbf{C}(z)$ as $\mathbf{C}(z) = [\mathbf{C}_0\ \mathbf{C}_1(z)]$ where $\mathbf{C}_0$ is an $N \times M$ lower triangular Toeplitz matrix. It follows that

$$\mathbf{C}(z)\mathbf{G} = \mathbf{C}_0\mathbf{G}_0.$$

Now the condition for perfect reconstruction becomes $\mathbf{S}\mathbf{C}_0\mathbf{G}_0 = \mathbf{I}$; that is, $\mathbf{S}$ should be an left inverse of the constant matrix $\mathbf{C}_0\mathbf{G}_0$. Using singular value decomposition (SVD), we can decompose $\mathbf{C}_0$ as,

$$\mathbf{C}_0 = \underbrace{[\mathbf{U}_0\mathbf{U}_1]}_{\mathbf{U}} \begin{pmatrix} \mathbf{\Lambda} \\ \mathbf{0} \end{pmatrix}_{N \times M} \mathbf{V}^T = \mathbf{U}_0\mathbf{\Lambda}\mathbf{V}^T, \tag{4}$$

where $\mathbf{U}$ and $\mathbf{V}$ are $N \times N$ and $M \times M$ unitary matrices. The column vectors of $\mathbf{U}$ are the eigenvectors of $\mathbf{C}_0\mathbf{C}_0^T$ and the column vectors of $\mathbf{V}$ are the eigenvectors of $\mathbf{C}_0^T\mathbf{C}_0$. The matrice $\mathbf{\Lambda}$ is diagonal and the diagonal elements λ_k are the singular values of $\mathbf{C}_0$, which are nonzero as $\mathbf{C}_0$ has full rank. The SVD of $\mathbf{C}_0$ immediately gives us one possible choice of $\mathbf{S}$,

$$\mathbf{S} = \mathbf{G}_0^T\mathbf{V}\mathbf{\Lambda}^{-1}\mathbf{U}_0^T. \tag{5}$$

However the above equation gives only one possible solution. To obtain all solutions, we note that the PR condition $\mathbf{S}\mathbf{C}_0\mathbf{G}_0 = \mathbf{I}$ only requires that $\mathbf{S}$ be a left inverse of $\mathbf{C}_0\mathbf{G}_0$. As $\mathbf{C}_0\mathbf{G}_0$ is of dimensions $N \times M$, the receiver $\mathbf{S}$ is not unique. In fact, we can choose $\mathbf{S} = \mathbf{G}_0^T\mathbf{V}\mathbf{\Lambda}^{-1}\left(\mathbf{I}\quad\mathbf{A}\right)\mathbf{U}^T$, where $\mathbf{A}$ is an arbitrary $M \times L$ matrix. The flexibility can be exploited to improve the frequency selectivity of the receiving filters or to minimize the total output noise power [7]. The discussion of the later is given next.

MMSE receiver

When the DMT system is perfect, the output noise comes entirely from the channel noise. We define the output average noise power E_N as $E_N = 1/M\sum_{k=0}^{M-1}\hat{\sigma}_{e_k}^2$. Then the optimal choice of $\mathbf{A}$ that minimizes E_N is given by [7]

$$\mathbf{A} = -\mathbf{U}_0^T\mathbf{R}_{ee}\mathbf{U}_1(\mathbf{U}_1^T\mathbf{R}_{ee}\mathbf{U}_1)^{-1}.$$

For the same transmitter $\mathbf{G}_0 = \mathbf{V}$, Fig. 5 shows the reduction in noise power when $\mathbf{A}$ is introduced. The channel used in this example is $C(z) = 1 + \rho z^{-1}$ and the noise source is the NEXT dominated crosstalk [1].

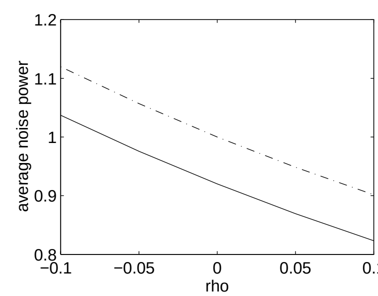

Figure 5: The average noise power as a function of ρ for the channel $1 - \rho z^{-1}$. The solid line is the average noise power of the MMSE receiver.

5. OPTIMAL DMT SYSTEM

We first derive the bit allocation formula for the generalized DMT system such that the transmitting power can be minimized for a given bit rate. Then we show how to design the optimal transceiver for arbitrary colored noise.

Let the bit rate in the k-th channel be b_k, then the total average bit rate is $b = \frac{1}{M}\sum_{k=0}^{M-1} b_k$. The input power of the k-th channel is σ_x^2, which is also the output signal power of the k-th channel at the receiver end due to the PR property. Suppose the output noise power of the kth channel is $\hat{\sigma}_{e_k}^2$. For most modulation systems under high bit rate assumption, we have

$\sigma^2_{x_k} 2^{-2b_k} = c\hat{\sigma}^2_{e_k}$, where the constant c depends on the given probability of symbol error P_e. Define $P(b)$ as the transmitting power needed for transmitting b bits. Then it can be shown that [7]

$$P(b) \geq c 2^{2b} E_0^{1/M}, \text{ where } E_0 = \Pi_{k=0}^{M-1} \hat{\sigma}^2_{e_k} \qquad (6)$$

The equality holds if and only if the bits are optimally allocated according to

$$b_k = b - \log \hat{\sigma}_{e_k} + \log E_0 / 2M. \qquad (7)$$

Let us define the coding gain $\mathcal{CG}$ as $P_{direct}(b)$, the power needed for transmitting b bits when there is no bit allocation, over $P(b)$. Without bit allocation, $b_k = b$, for $k = 0, 1, \cdots M-1$; $P_{direct}(b) = c 2^{2b} \sum_{k=0}^{M-1} \hat{\sigma}^2_{e_k}$. The coding gain of bit allocation is

$$\mathcal{CG} = \frac{1}{M} \sum_{k=0}^{M-1} \hat{\sigma}^2_{e_k} / \left(\Pi_{k=0}^{M-1} \hat{\sigma}^2_{e_k}\right)^{1/M} \geq 1.$$

The above inequality follows from the arithmetic mean over the geometric mean inequality.

From Fig. 6 we see the last part of the transmitter is the unitary matrix $\mathbf{G}_0^T \mathbf{V}$. Let us call it $\mathbf{Q}$. It can be shown that the optimal $\mathbf{Q}$ is determined by the autocorrelation matrix $\widetilde{\mathbf{R}}_{ee}$ of $\widetilde{\mathbf{e}}$, which is as indicated in Fig. 6. In particular the coding gain $\mathcal{CG}$ is maximized if $\mathbf{Q}$ is the KLT matrix for $\widetilde{\mathbf{R}}_{ee}$, i.e., $\mathbf{Q}^T \widetilde{\mathbf{R}}_{ee} \mathbf{Q}$ is a diagonal matrix [7]. The maximized coding gain is $\mathcal{CG}_{max} = \frac{1}{M} tr(\widetilde{\mathbf{R}}_{ee}) / \left(\det \widetilde{\mathbf{R}}_{ee}\right)^{1/M}$.

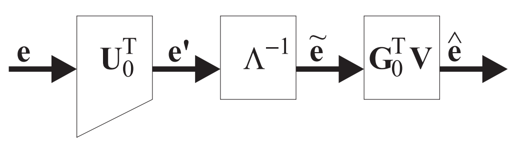

Figure 6: Block diagram of the receiver.

For the same probability of error and same bit rate, Fig. 7 shows $P_{opt}(b, M)/P_{DFT}(b, M)$, the ratio of power needed in optimal system over the power needed in the DFT-based system. We plot the ratio as a function of M for two different noise sources, the NEXT dominated noise and AWGN. The channel used in this example is $C(z) = 1 + 0.5z^{-1}$.

AWGN channels

When the channel noise is a white process, the autorrelation matrix $\mathbf{R}_{ee} = \sigma_e^2 \mathbf{I}$ is a diagonal matrix. The optimal $\mathbf{G}_0$ is simply $\mathbf{G}_0 = \mathbf{V}$. (This optimal solution of $\mathbf{G}_0$ in this case is consistent with what Kasturia et. al have obtained for AWGN channels from the view point of multidimentional signal constellations.) The

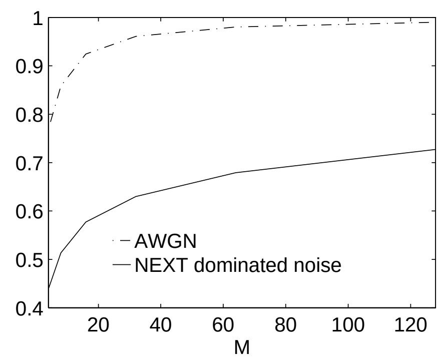

Figure 7: The ratio of the power needed in DFT based DMT system over the power needed in the optimal system for the same probability of error and the same bit rate.

coding gain is

$$\mathcal{CG}_{max} = \frac{\frac{1}{M}\sum_{k=0}^{M-1} 1/\lambda_k^2}{\left(\Pi_{k=0}^{M-1} 1/\lambda_k^2\right)^{1/M}}.$$

Although the DFT based DMT system are not optimal in general, it is asymptotically optimal for AWGN fading channel. In particular, for a given error probability and bit rate, the power required for transmitting b bits in DFT based DMT system becomes very close to the power required in the optimal system when M is sufficiently large [7]. From Fig. 7 we can see that for the AWGN channel, the ratio $P_{opt}(b, M)/P_{DFT}(b, M)$ approaches unity as M increases.

6. REFERENCES

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