CROSS-PRODUCT ALGORITHMS FOR SOURCE TRACKING USING AN EM VECTOR SENSOR

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ABSTRACT

We present two adaptive cross-product algorithms for tracking the direction to a moving source using an electromagnetic vector sensor. The first is a cross-product algorithm with a forgetting factor, for which we analyze the performance and derive an asymptotic expression of the variance of angular estimation error. We find the optimal forgetting factor that minimizes this variance. The second is a Kalman filter combined with the cross-product algorithm, which is applicable when the angular acceleration of the source is approximately constant.

1. INTRODUCTION

In this paper we develop two adaptive cross-product algorithms for tracking the direction to an electromagnetic (EM) source. These algorithms use measurements from an EM vector sensor (a device measuring the complete 6 components of an EM field at a single point). They extend the original method for stationary sources in [1], [2].

Inspired by the Poynting theorem, the algorithm in [1], [2] forms the cross-product of the electric field vector with the complex conjugate of the magnetic field vector and averages over time. The vector result is normalized to have unit length, yielding the estimate of the unit vector in the source direction. The resulting cross-product algorithm has no scalar-sensor counterpart. Its principal advantages and capabilities are the following:

- Very low computational complexity since no cost function minimization is needed.
- Ability to easily and equally work with sources of various types, such as wideband or narrowband signals, polarized or unpolarized.

The ability to work with wideband sources with low computational complexity is because the steering vector is not a function of the frequency. Other advantages that are inherited by the properties of the vector sensor are:

- Only one EM vector sensor is needed to track the source in 3D space, while occupying very little space.
- No need for sensor location calibration and time synchronization among different components, since no time delays are used.
- Isotropic response.

The paper is organized as follows. In Section 2 we introduce and analyze the simplest form of the adaptive cross-product algorithm, in which a forgetting factor is used to discount old data measurements in the averaging. For its performance analysis, a most difficult tracking scenario is considered where the signal angle of arrival has independent random Gaussian distributed increments. In Section 3 we consider a different scenario, where it is assumed that the angular velocity or angular acceleration of the angle of arrival is approximately constant. This is a more realistic setup and the tracking can be performed with a higher accuracy using a Kalman filter combined with the cross-product method. In [3], we illustrate the performance of the proposed algorithms via numerical examples. Section 5 summarizes our conclusions.

2. ANALYSIS

The cross-product algorithm for estimating the direction to a far-field source is based on the fact that in an electromagnetic plane wave the instantaneous vectors of the electric and magnetic fields and the direction vector of wave propagation are mutually orthogonal. Thus, if the former two vectors are measured by a 6-component vector sensor, the direction of the wave can be found by computing the cross product of these vectors.

Since in general the measurements are noisy and the signal is non-stationary, we propose to estimate the instantaneous vector of direction of the wave as a weighted average of the sequence of the individual cross products using an exponential window with a forgetting factor λ,

$$\hat{\boldsymbol{s}}_N = \frac{1}{\sum_{t=1}^{N} \lambda^{-t}} \sum_{t=1}^{N} \lambda^{-t} \mathrm{Re}\{\boldsymbol{y}_{\mathrm{E}}(t) \times \bar{\boldsymbol{y}}_{\mathrm{H}}(t)\}, \quad (2.1)$$

$$\hat{\boldsymbol{u}}_N = \hat{\boldsymbol{s}}_N / \|\hat{\boldsymbol{s}}_N\| \, . \quad (2.2)$$

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In (2.1), $\boldsymbol{y}_{\mathrm{E}}(t)$ is the 3D electric field measurement and $\bar{\boldsymbol{y}}_{\mathrm{H}}(t)$ the complex conjugate of the 3D magnetic field measurement, in phasor (complex envelope) form. We analyze, along the following lines, the performance of the algorithm in (2.1), (2.2) for tracking a nonstationary target.

Let

$$\hat{\boldsymbol{z}}_t = \mathrm{Re}\{\boldsymbol{y}_{\mathrm{E}}(t) \times \bar{\boldsymbol{y}}_{\mathrm{H}}(t)\}, \qquad t = 0, 1, \ldots, N \tag{2.3}$$

and let $\boldsymbol{e}_{\mathrm{E}}(t)$ and $\boldsymbol{e}_{\mathrm{H}}(t)$ be errors (an additive noise) that enter the measurements of the true electric and magnetic fields, $\boldsymbol{y}_{\mathrm{E}}(t)$ and $\boldsymbol{y}_{\mathrm{H}}(t)$. As in [2] we assume that these errors are zero mean, independent each of other and of the signal itself, and have the following covariances,

$$\begin{aligned}
\mathrm{E}\begin{bmatrix} \boldsymbol{e}_{\mathrm{E}}(t) \\ \boldsymbol{e}_{\mathrm{H}}(t) \end{bmatrix} [\bar{\boldsymbol{e}}_{\mathrm{E}}(s), \bar{\boldsymbol{e}}_{\mathrm{H}}(s)] &= \begin{bmatrix} \sigma_{\mathrm{E}}^2 I_3 & 0 \\ 0 & \sigma_{\mathrm{H}}^2 I_3 \end{bmatrix} && (2.4) \\
\mathrm{E}\begin{bmatrix} \boldsymbol{e}_{\mathrm{E}}(t) \\ \boldsymbol{e}_{\mathrm{H}}(t) \end{bmatrix} [\boldsymbol{e}_{\mathrm{E}}^T(s), \boldsymbol{e}_{\mathrm{H}}^T(s)] &= 0 \ \text{(for all } t \text{ and } s\text{)}. && (2.5)
\end{aligned}$$

In (2.5), the superscript "T" denotes the transpose and I_3 the 3D identity matrix. It is shown in [2] that $\hat{\boldsymbol{z}}_t$ can be written as

$$\hat{\boldsymbol{z}}_t = \boldsymbol{z}_t + \delta \boldsymbol{z}_t \tag{2.6}$$

where

$$\boldsymbol{z}_t = \sigma_s^2 \cdot \boldsymbol{u}_t \tag{2.7}$$

and $\sigma_s^2 = \mathrm{E}[|s(t)|^2]$ is the variance of the complex envelope of the (scalar) transmitted signal $s(t)$ (cf. eqn's (2.13) and (4.24) in [2]), and $\{\delta \boldsymbol{z}_t\}$ is a sequence of pairwise independent zero-mean random vectors. Note that the signal variance σ_s^2 may also depend on time, but in our first-order approximation this will not affect the analysis. Further it is assumed that the signal envelope $s(t)$ is statistically independent of $\boldsymbol{e}_{\mathrm{E}}(t)$ and $\boldsymbol{e}_{\mathrm{H}}(t)$ and has finite fourth-order moments.

The covariance matrix of $\delta \boldsymbol{z}_t$ is computed on the bottom of [2, p. 396]. This matrix depends on t through the instantaneous signal parameters, in particular the vector $\boldsymbol{u}_t$ and also the electromagnetic wave polarization. In this paper we need only the following expression, which is independent of these parameters, see [2, p. 396],

$$\mathrm{tr}\left\{(I - \boldsymbol{u}_t \boldsymbol{u}_t^T)\mathrm{cov}[\delta \boldsymbol{z}_t]\right\} = \frac{1}{2}(\sigma_{\mathrm{E}}^2 + \sigma_{\mathrm{H}}^2)\sigma_s^2 + 2\sigma_{\mathrm{E}}^2\sigma_{\mathrm{H}}^2. \tag{2.8}$$

Since for any matrices A, B with compatible dimensions it holds that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and $(I - \boldsymbol{u}_t\boldsymbol{u}_t^T)^2 = (I - \boldsymbol{u}_t\boldsymbol{u}_t^T)$, it also holds that

$$\begin{aligned}
&\mathrm{tr}\left\{(I - \boldsymbol{u}_t \boldsymbol{u}_t^T)\mathrm{cov}[\delta \boldsymbol{z}_t](I - \boldsymbol{u}_t \boldsymbol{u}_t^T)\right\} = \\
&= \frac{1}{2}(\sigma_{\mathrm{E}}^2 + \sigma_{\mathrm{H}}^2)\sigma_s^2 + 2\sigma_{\mathrm{E}}^2\sigma_{\mathrm{H}}^2 \ . && (2.9)
\end{aligned}$$

Note that the expression in (2.9) is the trace of the covariance matrix of $\tilde{\delta}\boldsymbol{z}_t \triangleq (I - \boldsymbol{u}_t\boldsymbol{u}_t^T)\delta\boldsymbol{z}_t$. The relation (2.9) will be used in the sequel.

Let Δ denote the forward difference operator, e.g.

$$\Delta \boldsymbol{u}_t = \boldsymbol{u}_{t+1} - \boldsymbol{u}_t \ , \tag{2.10}$$

and let

$$\tilde{\Delta}\boldsymbol{u}_t = (I_3 - \boldsymbol{u}_t\boldsymbol{u}_t^T)\Delta\boldsymbol{u}_t \ . \tag{2.11}$$

For analyzing the tracking, we shall consider a worst case model of the evolution of $\boldsymbol{u}_t$,

$$\boldsymbol{u}_{t+1} = \frac{\boldsymbol{u}_t + \boldsymbol{n}_t}{\|\boldsymbol{u}_t + \boldsymbol{n}_t\|} \tag{2.12}$$

where $\{\boldsymbol{n}_t\}$ are independent samples from the distribution $\mathcal{N}(0, \sigma_n^2 I_3)$. Thus, the source moves randomly and equally likely in all directions from its position at the previous time sample. For use in the later analysis we shall derive an approximate expression for the trace of the covariance matrix of $\tilde{\Delta}\boldsymbol{u}_t$. Using the Taylor series expansion of (2.12) we get

$$\boldsymbol{u}_{t+1} = \boldsymbol{u}_t + [I_3 - \boldsymbol{u}_t\boldsymbol{u}_t^T]\boldsymbol{n}_t + O_{\mathrm{UB}}(\|\boldsymbol{n}_t\|^2) \tag{2.13}$$

where $O_{\mathrm{UB}}(\|\boldsymbol{n}_t\|^2)$ stands for a remainder which is uniformly bounded in norm by a constant times $\|\boldsymbol{n}_t\|^2$ both for all $\|\boldsymbol{n}_t\| \le 1/2$ and for all $\|\boldsymbol{n}_t\| > 1/2$ (see proof of Lemma H.2 in [2]). Hence

$$\begin{aligned}
\mathrm{tr}\{\mathrm{cov}[\tilde{\Delta}\boldsymbol{u}_t]\} &= \mathrm{tr}\{\mathrm{cov}[(I_3 - \boldsymbol{u}_t\boldsymbol{u}_t^T)\boldsymbol{n}_t]\} + \mathrm{E}\{O_{\mathrm{UB}}(\|\boldsymbol{n}_t\|^3)\} \\
&= \mathrm{tr}[(I_3 - \boldsymbol{u}_t\boldsymbol{u}_t^T)\mathrm{cov}[\boldsymbol{n}_t](I_3 - \boldsymbol{u}_t\boldsymbol{u}_t^T)] + O(\sigma_n^3) \\
&= \sigma_n^2 \mathrm{tr}(I_3 - \boldsymbol{u}_t\boldsymbol{u}_t^T) + O(\sigma_n^3) \\
&= 2\sigma_n^2 + O(\sigma_n^3). && (2.14)
\end{aligned}$$

Let δ denote the estimation error operator, e.g.

$$\begin{aligned}
\delta \boldsymbol{u}_t &= \hat{\boldsymbol{u}}_t - \boldsymbol{u}_t && (2.15) \\
\delta \boldsymbol{s}_t &= \hat{\boldsymbol{s}}_t - \boldsymbol{z}_t && (2.16)
\end{aligned}$$

where we assume that $\hat{\boldsymbol{s}}_t$ in (2.1) is an estimate of the quantity $\boldsymbol{z}_t$ in (2.7) and put

$$\tilde{\delta}\boldsymbol{s}_t = (I_3 - \boldsymbol{u}_t\boldsymbol{u}_t^T)\delta\boldsymbol{s}_t \ . \tag{2.17}$$

Then using (2.2), (2.7), and a series expansion similar to those in (2.12) and (2.13) we obtain

$$\begin{aligned}
\hat{\boldsymbol{u}}_t &= \frac{\hat{\boldsymbol{s}}_t}{\|\hat{\boldsymbol{s}}_t\|} = \frac{\boldsymbol{z}_t + \delta\boldsymbol{s}_t}{\|\boldsymbol{z}_t + \delta\boldsymbol{s}_t\|} = \frac{\boldsymbol{u}_t + \delta\boldsymbol{s}_t\sigma_s^{-2}}{\|\boldsymbol{u}_t + \delta\boldsymbol{s}_t\sigma_s^{-2}\|} \\
&= \boldsymbol{u}_t + [I_3 - \boldsymbol{u}_t\boldsymbol{u}_t^T]\delta\boldsymbol{s}_t\sigma_s^{-2} + O_{\mathrm{UB}}(\|\delta\boldsymbol{s}_t\|^2\sigma_s^{-4}) \\
\delta\boldsymbol{u}_t &= \tilde{\delta}\boldsymbol{s}_t\sigma_s^{-2} + O_{\mathrm{UB}}(\|\delta\boldsymbol{s}_t\|^2\sigma_s^{-4}) && (2.18)
\end{aligned}$$

The angular estimation error can be approximated by

$$\delta\varphi_t = 2\arcsin(\|\hat{\boldsymbol{u}}_t - \boldsymbol{u}_t\|/2) = \|\delta\boldsymbol{u}_t\| + O_{\mathrm{UB}}(\|\delta\boldsymbol{u}_t\|^3) \tag{2.19}$$

Using (2.18), (2.19) and the fact that the absolute value of $\delta\varphi_t$ is bounded by a constant (equal to π) we get

$$\begin{aligned}
\delta\varphi_t &= \|\tilde{\delta}\boldsymbol{s}_t\|\sigma_s^{-2} + O_{\mathrm{UB}}(\|\delta\boldsymbol{s}_t\|^2\sigma_s^{-4}) && (2.20) \\
[\delta\varphi_t]^2 &= \|\tilde{\delta}\boldsymbol{s}_t\|^2\sigma_s^{-4} + O_{\mathrm{UB}}(\|\delta\boldsymbol{s}_t\|^3\sigma_s^{-6}) \ . && (2.21)
\end{aligned}$$

Hence, provided that the errors $\boldsymbol{e}_{\mathrm{E}}$, $\boldsymbol{e}_{\mathrm{H}}$ and $\delta\boldsymbol{s}_t$ have finite third-order moments, it holds

$$\mathrm{var}[\delta\varphi_t] = \mathrm{tr}\{\mathrm{cov}[\tilde{\delta}\boldsymbol{s}_t]\}\sigma_s^{-4} + O(\mathrm{E}[\|\tilde{\delta}\boldsymbol{s}_t\|^3]\sigma_s^{-6}) \ . \tag{2.22}$$

Finally note that $\hat{\boldsymbol{s}}_t$ in (2.1) can be written recursively as

$$\hat{\boldsymbol{s}}_t = \lambda \hat{\boldsymbol{s}}_{t-1} + (1-\lambda)\hat{\boldsymbol{z}}_t \ . \tag{2.23}$$

The error of $\hat{\boldsymbol{s}}_t$ is

$$\begin{aligned} \delta \boldsymbol{s}_t &= \hat{\boldsymbol{s}}_t - \boldsymbol{z}_t = \lambda \hat{\boldsymbol{s}}_{t-1} + (1-\lambda)\hat{\boldsymbol{z}}_t - \boldsymbol{z}_t \\ &= \lambda(\boldsymbol{z}_{t-1} + \delta \boldsymbol{s}_{t-1}) + (1-\lambda)(\boldsymbol{z}_t + \delta \boldsymbol{z}_t) - \boldsymbol{z}_t \\ &= \lambda \delta \boldsymbol{s}_{t-1} + (1-\lambda)\delta \boldsymbol{z}_t - \lambda \Delta \boldsymbol{z}_t \ . \end{aligned} \tag{2.24}$$

Multiplying the last equality by $I_3 - \boldsymbol{u}_t \boldsymbol{u}_t^T$ and neglecting higher than first order error terms we get

$$\tilde{\delta} \boldsymbol{s}_t = \lambda \tilde{\delta} \boldsymbol{s}_{t-1} + (1-\lambda)\tilde{\delta} \boldsymbol{z}_t - \lambda \tilde{\Delta} \boldsymbol{z}_t \ . \tag{2.25}$$

Thus, the error sequence $\{\tilde{\delta}\boldsymbol{s}_t\}$ can be obtained by the following linear filtering of the sequencies $\{\tilde{\delta}\boldsymbol{z}_t\}$ and $\{\tilde{\Delta}\boldsymbol{z}_t\}$,

$$\begin{aligned} \tilde{\delta} \boldsymbol{s}_t &= \frac{1-\lambda}{1-\lambda q^{-1}}\, \tilde{\delta} \boldsymbol{z}_t + \frac{-\lambda}{1-\lambda q^{-1}}\, \tilde{\Delta} \boldsymbol{z}_t \\ &\stackrel{\triangle}{=} \Phi_1(q^{-1})\, \tilde{\delta} \boldsymbol{z}_t + \Phi_2(q^{-1})\, \tilde{\Delta} \boldsymbol{z}_t \ . \end{aligned} \tag{2.26}$$

where q^{-1} is the backward time shift operator.

Then, in the limit $N \to \infty$ it holds that

$$\begin{aligned} \mathrm{tr}\{\mathrm{cov}[\tilde{\delta}\boldsymbol{s}_t]\} &= \frac{1}{2\pi i} \oint \Phi_1(z)\, \Phi_1(z^{-1})\, z^{-1}\, dz \cdot \mathrm{tr}\{\mathrm{cov}[\tilde{\delta}\boldsymbol{z}_t]\} \\ &+ \frac{1}{2\pi i} \oint \Phi_2(z)\, \Phi_2(z^{-1})\, z^{-1}\, dz \cdot \mathrm{tr}\{\mathrm{cov}[\tilde{\Delta}\boldsymbol{z}_t]\} \end{aligned}$$

where the integration proceeds along the unit circle in the complex plane. After a straightforward calculation we get

$$\mathrm{tr}\{\mathrm{cov}[\tilde{\delta}\boldsymbol{s}_t]\} = \frac{1-\lambda}{1+\lambda}\, \mathrm{tr}\{\mathrm{cov}[\tilde{\delta}\boldsymbol{z}_t]\} + \frac{\lambda^2}{1-\lambda^2}\, \mathrm{tr}\{\mathrm{cov}[\tilde{\Delta}\boldsymbol{z}_t]\} \ . \tag{2.27}$$

Now, from (2.7) it follows that

$$\begin{aligned} \Delta \boldsymbol{z}_t &= \boldsymbol{z}_{t+1} - \boldsymbol{z}_t = \sigma_s^2 \Delta \boldsymbol{u}_t + \Delta \sigma_s^2 \boldsymbol{u}_t \\ \tilde{\Delta} \boldsymbol{z}_t &= (I_3 - \boldsymbol{u}_t \boldsymbol{u}_t^T)\Delta \boldsymbol{z}_t = \\ &= \sigma_s^2 (I_3 - \boldsymbol{u}_t \boldsymbol{u}_t^T)\Delta \boldsymbol{u}_t = \sigma_s^2 \tilde{\Delta} \boldsymbol{u}_t . \end{aligned} \tag{2.28}$$

and

$$\mathrm{cov}[\tilde{\Delta}\boldsymbol{z}_t] = \sigma_s^4 \mathrm{cov}[\tilde{\Delta}\boldsymbol{u}_t] \ . \tag{2.29}$$

Substituting (2.29), (2.27), (2.14) and (2.9) into (2.22) we obtain

$$\begin{aligned} \mathrm{var}[\delta\varphi_t] &= \frac{1-\lambda}{1+\lambda}\, \sigma_x^2 + \frac{2\lambda^2}{1-\lambda^2}\, \sigma_n^2 + \\ &+ O(\mathrm{E}[\|\tilde{\delta}\boldsymbol{z}_t\|^3]\sigma_s^{-6} + \sigma_n^3) \end{aligned} \tag{2.30}$$

where

$$\sigma_x^2 = \frac{\sigma_{\mathrm{E}}^2 + \sigma_{\mathrm{H}}^2}{2\sigma_s^2} + \frac{2\sigma_{\mathrm{E}}^2 \sigma_{\mathrm{H}}^2}{\sigma_s^4} \ . \tag{2.31}$$

We use the above result to compute an optimal value of the forgetting factor. If the remainder term in (2.30) is neglected, the leading term in the equation is minimized for

$$\lambda = \lambda_0 \stackrel{\triangle}{=} 1 + \kappa - \sqrt{2\kappa + \kappa^2} \tag{2.32}$$

where

$$\kappa = \frac{\sigma_n^2}{\sigma_x^2} \ . \tag{2.33}$$

The Taylor series expansion of (2.32) gives

$$\lambda_0 = \begin{cases} 1 - \sqrt{2\kappa} + O(\kappa) & \text{for} \quad \kappa \to 0 \\ \frac{1}{2\kappa} + O(\frac{1}{\kappa^2}) & \text{for} \quad \kappa \to \infty \end{cases} \ . \tag{2.34}$$

This optimal choice of λ is a trade-off between system responsiveness and noise sensitivity. The source position fluctuates with amplitude that increases with σ_n, so when σ_n^2 is large we expect a smaller forgetting factor to yield better performance. A smaller forgetting factor lets old data samples be "forgotten" more quickly. But as with other tracking systems, a responsive system is generally more susceptible to measurement noise, since time averages are generally short, while a noise tolerant system is not able to follow rapid movement of the source. Hence a trade-off between these conflicting performance measures is needed and is solved by λ_0.

3. TRACKING OF DRIFTING TARGETS

In cases when the targets are subject to a slow and persistent drift or a slow and persistent acceleration, the above algorithm exhibits an estimation delay. This delay can be eliminated by an algorithm which, in addition to the instantaneous direction vector, estimates also the angular velocity and acceleration of the target. The algorithm proposed in this section is based on a modified Kalman filter and is inspired by a similar one in [4].

Let $\dot{\boldsymbol{u}}_t$ and $\ddot{\boldsymbol{u}}_t$ be the bearing velocity and acceleration, respectively, i.e. the first and the second time derivative of the direction vector $\boldsymbol{u}_t$. Since $\boldsymbol{u}_t$ has a unit norm, $\dot{\boldsymbol{u}}_t$ and $\ddot{\boldsymbol{u}}_t$ are orthogonal to this vector.

Introduce the notation

$$\boldsymbol{\theta}_t = \begin{bmatrix} \boldsymbol{u}_t \\ \dot{\boldsymbol{u}}_t \\ \ddot{\boldsymbol{u}}_t \end{bmatrix} \tag{3.35}$$

and the function

$$\mathbf{orthnorm}\left(\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{x}_1/\|\boldsymbol{x}_1\| \\ (I_3 - \boldsymbol{x}_1\boldsymbol{x}_1^T/\|\boldsymbol{x}_1\|^2)\boldsymbol{x}_2 \\ (I_3 - \boldsymbol{x}_1\boldsymbol{x}_1^T/\|\boldsymbol{x}_1\|^2)\boldsymbol{x}_3 \end{bmatrix} \tag{3.36}$$

which is the projection mapping into the set of properly normalized vectors $\boldsymbol{\theta}$. Specifically, The function $\mathbf{orthnorm}(\cdot)$ is used to assure that the vector of direction (on top) has a unit norm and that the velocity and acceleration vectors (below) are orthogonal to it.

Assume that time evolution of $\boldsymbol{u}_t$ is governed by the equation

$$\boldsymbol{\theta}_{t+1} = \mathbf{orthnorm}(F\boldsymbol{\theta}_t + w_t) \tag{3.37}$$

where

$$F = \begin{bmatrix} I_3 & TI_3 & \frac{T^2}{2} I_3 \\ 0_{3\times 3} & I_3 & TI_3 \\ 0_{3\times 3} & 0_{3\times 3} & I_3 \end{bmatrix} \tag{3.38}$$

T denotes the sampling interval (in seconds), $\{w_t\}$ is a random process noise term that accounts for random perturbations about the constant acceleration trajectories. Note that the model in (3.37) without the operation of orthonormalization is generally used in discussions of tracking problems and also in [4]. In this model, the acceleration is assumed to be constant during the sampling interval and may differ from one interval to another through random perturbations.

The observation is then the real part of the EM cross product:

$$\hat{\boldsymbol{z}}_t = \sigma_s^2 \cdot \boldsymbol{u}_t + \delta \boldsymbol{z}_t = H\boldsymbol{\theta}_t + \delta \boldsymbol{z}_t \tag{3.39}$$

where

$$H = \sigma_s^2 \left[I_3, 0_{3\times 6} \right] , \tag{3.40}$$

which is of dimension 3×9 and $0_{3\times 6}$ is a 3×6 dimension matrix with 0 entries. Assume that covariance matrices of w_t and of $\delta \boldsymbol{z}_t$, denoted $\boldsymbol{C}_{\delta w}$ and $\boldsymbol{C}_{\delta z}$, respectively, are known. Tracking of the state vector $\boldsymbol{\theta}_t$ can be solved by the following modified Kalman filter,

$$\begin{aligned}
\hat{\boldsymbol{\theta}}_{t|t} &= \textbf{orthnorm}\left(\hat{\boldsymbol{\theta}}_{t|t-1} + L_t[\hat{\boldsymbol{z}}_t - H\hat{\boldsymbol{\theta}}_{t|t-1}]\right) && (3.41)\\
\hat{\boldsymbol{\theta}}_{t+1|t} &= \textbf{orthnorm}\left(F\hat{\boldsymbol{\theta}}_{t|t}\right) && (3.42)\\
L_t &= \Sigma_{t|t-1}H^T\left[H\Sigma_{t|t-1}H^T + \boldsymbol{C}_{\delta z}\right]^{-1} && (3.43)\\
\Sigma_{t|t} &= (I_9 - L_t H)\Sigma_{t|t-1} && (3.44)\\
\Sigma_{t+1|t} &= F\Sigma_{t|t}F^T + \boldsymbol{C}_w \ . && (3.45)
\end{aligned}$$

This filter is obtained as a modification of a Kalman filter for the model in (3.37)–(3.39) excluding the orthonormalization in (3.37). Simulations show that the proposed filter works quite well even if the theoretical covariance matrices $\boldsymbol{C}_w$ and $\boldsymbol{C}_{\delta z}$ are replaced by identity matrices multiplied by appropriately chosen scalar factors.

Let $\boldsymbol{C}_w = \sigma_w^2 I_9$ and $\boldsymbol{C}_{\delta z} = \sigma_z^2 I_3$. Note from (3.43)–(3.45) that the asymptotic Kalman gain L_t for $t \to \infty$ does not depend on the variances σ_w^2 and σ_z^2 themselves but on their ratio $\xi = \sigma_w^2/\sigma_z^2$. Thus, it is possible to put $\sigma_z^2 = 1$ and control the asymptotic performance of the filter by a single parameter σ_w^2. Note, however, that if $\boldsymbol{C}_w$ and $\boldsymbol{C}_{\delta z}$ are not selected realistically but as *ad hoc* constants, then $\Sigma_{t|t}$ and $\Sigma_{t+1|t}$ in (3.44) and (3.45) also lose the interpretations of mean-square estimation errors.

It is worth noting that if all three of matrices $\boldsymbol{C}_w$, $\boldsymbol{C}_{\delta z}$ and $\Sigma_{1|0}$ are taken as multiples of the identity matrix, the Kalman gain is identical for all three components of vectors of the direction, velocity and acceleration. Hence it suffices to compute a gain vector according to (3.43)–(3.45) for only one of the components, so that e.g. the matrix to be inverted in (3.43) will have the size of only 3×3 instead of 9×9.

The price for the improved performance of the algorithm presented in this section, compared with those proposed in the previous section in scenarios with a persistent drift, is a higher computational complexity and also a longer transients at the beginning of the tracking, until the correct wave direction, velocity and acceleration are approached.

4. CONCLUDING REMARKS

We discussed two algorithms for tracking the direction to a moving source, using data from an EM vector sensor. We analyzed the performance of the cross-product algorithm with a forgetting factor, and computed an asymptotic expression of its variance of angular estimation error. This expression was used to find the optimal forgetting factor that minimizes the error variance, as a function of the source dynamics and sensor noise variances. The main advantage of this algorithm is in its computational efficiency. We then presented a Kalman fliter combined with the cross-product algorithm, which is applicable when the angular acceleration of the source is approximately constant. This method needs more computations than the cross-product algorithm, but is more accurate after convergence. In [3] we develop a third algorithm of multiple forgetting factors type, and uses a weighted average of the outputs of cross-product algorithms. It has a self tuning ability which is useful when no *a priori* knowledge of the source dynamics is available, but requires more computations then the single cross-product algorithm. We also present numerical examples demonstrating the performance of the 3 algorithms in different scenarios.

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