NEW RESULTS AND OPEN PROBLEMS ON NONUNIFORM FILTER-BANKS

Sony Akkarakaran and P.P. Vaidyanathan

Department of Electrical Engineering 136-93
California Institute of Technology
Pasadena, CA 91125 USA.
E-mail: sony@systems.caltech.edu, ppvnath@sys.caltech.edu

ABSTRACT

Nonuniform filter-banks (FB's) have traditionally been built either by cascading uniform ones in a tree structure or by direct design methods that lead to near-perfect reconstruction. However many theoretical issues remain unresolved. This paper begins by pointing out a number of these issues, and summarizes the known conditions for existence of nonuniform perfect reconstruction (PR) FB's. As a new contribution, we simplify some of these conditions and make them more explicit. We provide examples that illustrate some hitherto unobserved connections between these conditions.

1. INTRODUCTION

A nonuniform filter-bank (FB), shown in Fig. 1, is a FB in which the channel decimation factors n_k are not necessarily equal (unlike a uniform FB where they must be equal). This paper considers only maximally decimated FB's, i.e. we assume $\sum_k \frac{1}{n_k} = 1$. Given such a set of positive integers n_k, we can sometimes build a nonuniform FB with n_k as the decimation factors by cascading uniform FB's in the form of a tree structure. This is schematically illustrated in Fig. 2 for the set $\{3, 3, 6, 6\}$. Such a FB would hence be perfectly reconstructing (PR) if the cascaded uniform FB's were PR. One cannot always build nonuniform FB's in this way. However if we allow the filters in the FB to be ideal (unrealizable) then we can always construct a PR nonuniform FB. This is done by using ideal complex brickwall analysis and synthesis filters with bandwidths inversely proportional to the decimation factors, as shown in Fig. 3. With realizable filters, we can achieve approximate or 'near-perfect reconstruction' in several ways, as is shown in [3, 4] and in references therein. A more general situation where the decimators could be fractional has been considered in [5]. However many theoretical issues involved in obtaining *exact* reconstruction remain unresolved even when the decimators are integers. (These issues do not arise if near-PR designs are sufficient). This paper points out these issues, and summarizes the known conditions for existence of nonuniform PRFB's. The new results, some of which are listed as Assertions 1–5, increase the clarity of the known conditions, and show some new connections between them.

Work supported in parts by the NSF grant MIP 0703755.

One of the basic problems is as follows: Given integers n_k satisfying the maximal decimation condition

$$\sum_k \frac{1}{n_k} = 1,$$

find necessary and sufficient conditions on the set of n_k so that a PRFB can be built using n_k as the channel decimation factors. We can always build such a PRFB using ideal filters, as shown earlier. However the problem becomes complicated if we restrict the filters to be realizable (i.e. rational). Another unsolved problem is that of **parameterization of nonuniform PRFB's**: Suppose the decimators n_k allow building of rational PRFB's, find *all* possible rational PRFB's (FIR or IIR) that can be built.

Note that an obvious sufficient condition on the n_k for existence of realizable PRFB's is that all n_k be equal — the theory of rational *uniform* PRFB's (both FIR and IIR) is well understood. Further a cascade of PRFB's in a tree structure results in another PRFB. So another sufficient condition on the n_k is that they be obtainable from a tree structure in which each unit in itself obeys some sufficient condition. For example, if each unit is a set of equal integers then we can cascade uniform PRFB's to build the nonuniform one (as in Fig. 2). However, neither of these conditions is necessary. Also, even if the n_k can be derived from a tree, all PRFB's built using n_k as decimators need not in general be derivable from tree structures. In the special case when the n_k are derivable from a dyadic 'wavelet-tree', [6] shows that any orthonormal PRFB (i.e. one satisfying $F_k(e^{j\omega}) = H_k^*(e^{j\omega})$ in Fig. 1) must be derivable from the same tree. However the general problem of parameterizing all possible PRFB's with a given set of decimators is still unsolved. Thus, very little is known about general nonuniform PRFB's as against uniform ones.

2. DELAY CHAIN FILTER-BANKS

This section considers the case when all analysis filters are constrained to be delays (i.e. in Fig. 1, $H_k(z) = z^{-l_k}$ for some integers l_k). Delays are too simple filters to be practically very useful, but the situation is instructive in terms of conditions on the decimators for PR. An example of this case was considered in [1]. Here we make the condition more explicit, and use it to derive further conclusions. The condition is derived based on the realization that PR is possible iff **no input sample goes through more than one channel** [1] (in which case, maximal decimation ensures that each sample goes through exactly one channel). Thus we require $mn_i - l_i \neq qn_j - l_j$ when $i \neq j$, for every choice of integers m, q. If this requirement is satisfied, then PR is obtained with $H_k(z) = z^{-l_k}$ and $F_k(z) = z^{l_k}$. Such a PRFB in which all filters are delays is called a delay chain FB, and is thus necessarily orthonormal.

Rewriting the above requirement, we conclude the following: The necessary and sufficient condition on the set of n_i for existence of a PR delay chain FB with n_i as decimators is that **there exist a corresponding set of integers l_i satisfying the property that $l_i - l_j$ is not a multiple of $\gcd(n_i, n_j)$ if $i \neq j$.** In particular, all l_i have to be distinct. Also, since a delay of a multiple of n_i can be moved from the analysis filter across the decimator n_i (using the noble identities), we can assume $0 \leq l_i < n_i$ without loss of generality. The condition on the n_i can hence be tested by a finite procedure. We also note that in the testing, one of the delays can be chosen arbitrarily, e.g. $l_0 = 0$ without loss of generality. An example from [1] satisfying the condition is the set of 23 integers $\{6, 10, 15, 30, \ldots, 30\}$ (30 occuring 20 times). Since the gcd of these integers is unity, this set cannot be built from a tree structure of uniform FB's (as in Fig. 2). Thus the **tree structure condition is not necessary.** We now use the same example to show

Assertion 1. Existence of a delay chain PRFB does not imply its uniqueness: Both the sets of l_i $\{0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 13, 14, 15, 16, 19, 20, 22, 23, 25, 26, 27, 28, 29\}$ and $\{0, 5, 7, 1, 2, 3, 4, 8, 9, 10, 11, 13, 14, 16, 17, 19, 20, 21, 23, 26, 27, 28, 29\}$ can be verified to result in PR delay chains when used along with the above set of decimators. On the other hand, the uniform L-channel delay chain FB is unique if the delays satisfy $0 \leq l_i < L$. Thus, even with such simple filters (delays) in the nonuniform FB, we encounter issues that do not occur at all with uniform FB's. This gives an indication that the full parameterization of all nonuniform FB's is much more involved than that of uniform ones.

3. THE FIR AND RATIONAL CASES

Constraining the filters to be FIR or rational is much weaker than insisting that they be delays. However it is not known whether this allows us to relax the condition on the decimators as compared to the condition of the previous section. This section shows how the polyphase decomposition, a powerful tool for uniform FB parameterization, is not as useful if we try to apply it to nonuniform FB's. The next section describes the known necessary conditions on the decimators, and points out some relations between them.

We can attempt to study the the nonuniform FB of Fig. 1 by redrawing it as an equivalent uniform FB [1, 2, 5] with decimation factor $L = \mathrm{lcm}\{n_i\}$. The k-th channel in the nonuniform FB, with decimator n_k, is replaced by $p_k = L/n_k$ channels in the uniform one as shown in Fig. 4. On both analysis and synthesis sides, the filters in these p_k channels are delayed versions of each other, and thus have mutual dependencies. Equivalently, the analysis polyphase matrix $\mathbf{E}(z)$ of the large uniform FB has a special structure: Its rows can be partitioned into groups, where the k-th group, shown in Fig. 5, has p_k rows and corresponds to the analysis filter $H_k(z)$ in the original nonuniform system. The $E_l^k(z)$ ($l \in \{0, 1, \ldots, L-1\}$) in Fig. 5 are the L-th order polyphase components of the filter $H_k(z)$. The first row consists of these polyphase components. Subsequent rows are obtained by shifting length$-n_k$ blocks of the previous row to the right with the last block circulated back to the left end after multiplication by z^{-1}. (However $\mathbf{E}(z)$ is not a block pseudocirculant, because this construction is carried out separately for the groups of rows corresponding to different analysis filters $H_k(z)$). Similarly the synthesis polyphase matrix $\mathbf{R}(z)$ of the large uniform system has a form described by taking the transpose of the structure shown by Fig. 5 and replacing $z^{-1}$with z. Even though we can find all general matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ that can give PR, it is difficult to find (i.e. parameterize) all such matrices that have the group structure described above. There has been no reported method for exactly imposing this structure in the design of the large uniform FB.

If the matrix $\mathbf{E}(z)$ is paraunitary (PU), then the corresponding nonuniform FB is orthonormal. We can get PR in this case by taking the synthesis polyphase matrix of the uniform FB to be $\mathbf{R}(z) = \tilde{\mathbf{E}}(z)$, [1] or equivalently by taking $F_k(z) = \tilde{H_k}(z)$ in Fig. 1. If on the other hand, $\mathbf{E}(z)$ is not PU, then a PRFB can be built iff the inverse $\mathbf{R}(z) = \mathbf{E}^{-1}(z)$ has the special structure described above, so that it is derivable from a nonuniform synthesis bank. (This is automatically ensured if $\mathbf{E}(z)$ is PU, since in that case $\mathbf{R}(z) = \tilde{\mathbf{E}}(z)$). However we note

Assertion 2. If the nonuniform synthesis bank is allowed to have LPTV(L) (linear periodically time-varying with period L) systems in place of the filters $F_k(z)$ then we can always obtain PR for arbitrary $\{n_k\}$ (satisfying $\sum_k \frac{1}{n_k} = 1$). This is achieved by replacing $F_k(z)$ with a LPTV(L) system with $p_k = L/n_k$ components $F_k^{(i)}(z)$, $i = 0, 1, \ldots, p_k - 1$ as in Fig. 6. Comparison of Figs. 6 and 4 shows that the $F_k^{(i)}(z)$ act exactly like synthesis filters in the large uniform FB. So they can be chosen to make the system equivalent to the large uniform FB with $\mathbf{R}(z) = \mathbf{E}^{-1}(z)$.

However the condition on the n_k under which we can find LTI analysis and synthesis filters giving PR remains unknown. If we can find such filters then [1] shows that we can also find filters that further form an orthonormal FB. However this FB would in general have IIR filters even if the original filters were FIR. Again, existence issues for FIR orthonormal FB's with arbitrary $\{n_k\}$ remain unsolved.

4. NECESSARY CONDITIONS AND THEIR INTERRELATIONS

Some necessary conditions have been found on the decimators in order that rational filters giving PR exist. This section states them and shows some relations between them.

- *Compatibility test* [2]: This arises because each alias frequency in the output should occur at least twice for aliasing to be cancelled. The test is stated in [1, 2] as an algorithm to be performed on the decimator values, i.e. (1) Let the set of decimators be ordered $n_0 \leq n_1 \leq \ldots \leq n_{M-1}$, then if $n_{M-2} \neq n_{M-1}$ the set is not compatible; else (2) delete all decimators that are factors of n_{M-1} and repeat step (1) on

[1] $\tilde{\mathbf{E}}(z) = \mathbf{E}^{*T}(1/z^*)$

the new set. The starting set is compatible iff the algorithm runs till all decimators are deleted. We now note

Assertion 3. An equivalent and simpler restatement of the compatibility test is as follows: **Every decimator must be a factor of some other decimator**. A study of the algorithm of [1, 2] reveals that it is essentially checking this condition. The largest decimator can only be a factor of another equal decimator. So it must occur at least twice, as checked by Step (1) of the algorithm. Once all decimators that are factors of the largest one are removed (in Step (2)) then the remaining decimators cannot be factors of any of those that have been removed. So we can repeat Step (1) on the remaining set.

• *A stronger compatibility test:* If the decimators are denoted by distinct integers n_i with the decimator n_i occurring N_i times, then define positive integers $p_i = \frac{\text{lcm}\{n_i\}}{n_i}$ and $m_j = \min_{i \neq j} \frac{\text{lcm}(p_i, p_j)}{p_j}$. Then [1] shows that a necessary condition for PR is that $m_j - 1 < N_j$ for all j. Now from the definitions it follows that $m_j = 1$ iff some p_i is a factor of p_j, which in turn is iff n_j is a factor of n_i. Thus the earlier compatibility condition can be restated as follows: If $N_j = 1$ then $m_j = 1$ too. We see that this is automatically implied by the strong compatibility test. Further, the strong test is strictly stronger, i.e. it can be violated even though the compatibility condition holds, as shown in [1].

• *Non-coprime decimators:* Another necessary condition [1] is that **no two decimators be coprime.** This arises from the equivalence between PR and the biorthogonality relation $(H_i(z)F_l(z)) \downarrow_{g_{il}} = \delta(i - l)$ in Fig. 1. Here $g_{il} = \gcd(n_i, n_l)$, so coprime n_i and n_l would imply $g_{il} = 1$ and hence the filters $H_i(z)$ and $F_l(z)$ cannot both be rational.

Assertion 4. The strong compatibility condition and the condition on non-coprime decimators are unconnected, i.e. the fact that one holds does not imply anything about whether the other holds or not. This is proved by the sets $\{3, 5, 15, \ldots, 15\}$ (15 occuring 7 times) and $\{6, 6, 6, 6, 6, 8, 24\}$. The former has two coprime decimators (3 and 5) but can be verified to satisfy both the compatibility conditions. The latter on the other hand does not have a pair of coprime decimators, but violates both the compatibility conditions because the largest decimator 24 occurs only once.

Assertion 5. It is not known whether a situation is possible where a delay chain does not exist, but a FIR or IIR solution exists. For example, the set $\{6, 10, 15, \ldots, 15, 30, 30\}$ (15 occuring 10 times) cannot be used to build a delay chain PRFB – the condition of Section 2 is not satisfied. To verify this, we use the discussion and notation in Section 2. We set $l_0 = 0$ corresponding to $n_0 = 6$, and then find that all l_i corresponding to $n_i = 15$ are uniquely determined by the condition of Section 2. This leaves no satisfactory choice for l_1 corresponding to $n_1 = 10$, and hence the condition is violated. However this set of numbers satisfies all the necessary conditions of this section, and so it is not clear whether a FIR or IIR solution exists in this case.

5. CONCLUDING REMARKS

The example of Assertion 5 shows that it is not known whether taking together the unconnected necessary conditions of compatibility and non-coprimeness yields a sufficient condition for existence of rational nonuniform PRFB's. We may note that the set of n_k in this example was created from the one in Section 2 by replacing pairs of decimators of value 30 by decimators of value 15. In other words, cascading the example of Assertion 5 with appropriately placed two-channel uniform FB's will produce the example of Section 2. However if we go a step further and replace the last pair of 30's by 15 then the set loses the compatibility condition.

Lastly, existence of a delay chain obviously implies that of FIR and rational PRFB's (a delay chain is both FIR and rational). Indeed, the condition of Section 2 must hence imply all the necessary conditions of Section 4. (It might be instructive to try to prove this fact more directly.) The interesting question in this case is what other rational FB's exist with the same set of decimators. One way to create other PRFB's is by grouping together a set of channels with equal decimation ratios and inserting an invertible matrix after the decimators. It is always possible to do this because by compatibility test, at least 2 channels will have the same decimation. In general this method will *not* cover all possible FB's that can be built using the same set of decimators – simple example sets like $\{2, 4, 4\}$ demonstrate this point. However if we consider only sets that do not come from tree structures then it is not clear whether the method covers all possible FB's or not.

One possible approach towards solving the problem in its full generality is to try and characterize sets of integers satisfying the various combinations of necessary conditions. However since much more is known about sets that come from tree structures, we must eliminate these from consideration. It might help if we could give a simple characterization for the sets which can be generated by tree structures. For example, if the set of decimators contains only two distinct values, then one can show that it can always be built from a tree of uniform FB's.

6. REFERENCES

[1] I.Djokovic and P.P.Vaidyanathan, "Results on Biorthogonal Filter Banks," *ACHA,* pp. 329-343, 1994.

[2] P.-Q.Hoang and P.P.Vaidyanathan, "Non-uniform Multirate Filter Banks: Theory and Design," in *IEEE ISCAS,* Portland, Oregon, May 1989, pp.371-374.

[3] J.Li, T.Q.Nguyen, and S.Tantaratana, "A Simple Design Method for Near-Perfect-Reconstruction Nonuniform Filter Banks," *IEEE Trans. Signal Processing,* vol. 45, pp. 2105-2109, Aug. 1997.

[4] K.Nayebi, T.P.Barnwell,III, and M.Smith, "Nonuniform Filter Banks: A Reconstruction and Design Theory," *IEEE Trans. SP,* pp. 1114-1127, Mar. 1993.

[5] J.Kovacevic and M.Vetterli, "Perfect Reconstruction Filter Banks with Rational Sampling Factors," *IEEE Trans. SP,* vol. 41, pp. 2047-2066, June 1993.

[6] A.K.Soman and P.P.Vaidyanathan, "On Orthonormal Wavelets and Paraunitary Filter Banks," *IEEE Trans. Signal Processing,* vol. 41, pp. 1170-1183, Mar. 1993.

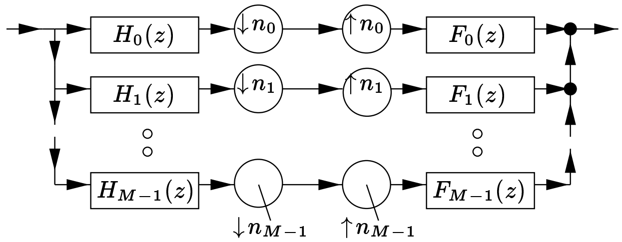

Fig. 1. A general nonuniform filter bank.

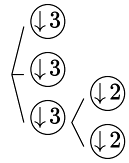

Fig. 2. Tree structure.

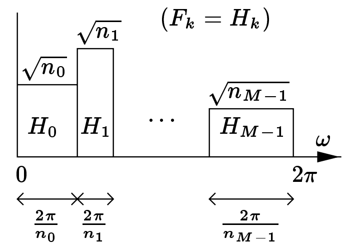

Fig. 3. Ideal nonuniform PRFB.

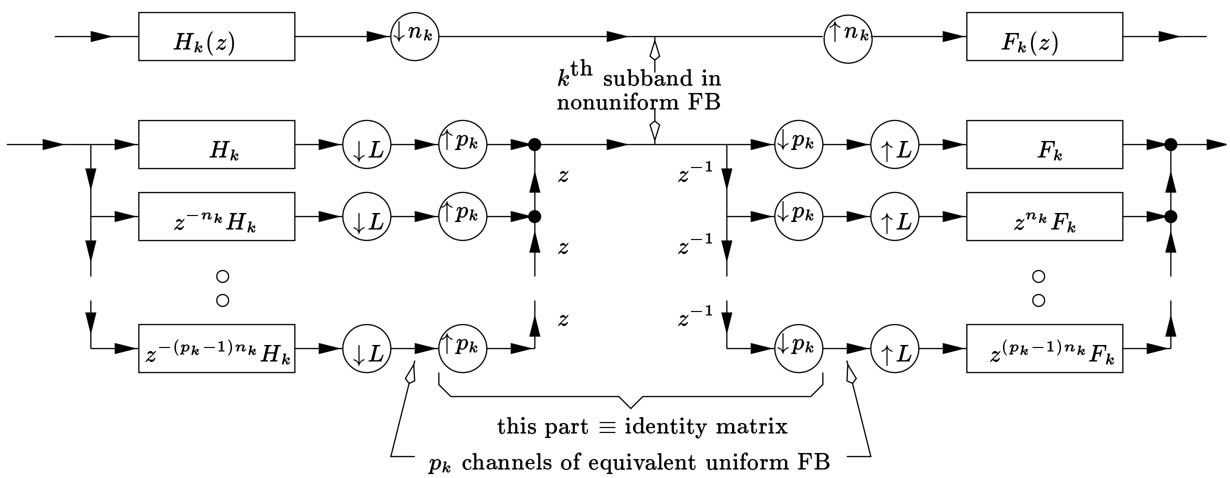

Fig. 4. Creating a uniform FB equivalent to a nonuniform one. $(L = \mathrm{lcm}\{n_i\} = n_k p_k)$

$$\left[\begin{array}{ccc|ccc|ccc}
E_0^k & \cdots & E^k_{n_k-1} & E^k_{n_k} & \cdots & E^k_{(p_k-1)n_k-1} & E^k_{(p_k-1)n_k} & \cdots & E^k_{L-1} \\
z^{-1}E^k_{(p_k-1)n_k} & \cdots & z^{-1}E^k_{L-1} & E^k_0 & \cdots & E^k_{(p_k-2)n_k-1} & E^k_{(p_k-2)n_k} & \cdots & E^k_{(p_k-1)n_k-1} \\
z^{-1}E^k_{(p_k-2)n_k} & \cdots & z^{-1}E^k_{(p_k-1)n_k-1} & z^{-1}E^k_{(p_k-1)n_k} & \cdots & E^k_{(p_k-3)n_k-1} & E^k_{(p_k-3)n_k} & \cdots & E^k_{(p_k-2)n_k-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
z^{-1}E^k_{n_k} & \cdots & z^{-1}E^k_{2n_k-1} & z^{-1}E^k_{2n_k} & \cdots & z^{-1}E^k_{L-1} & E^k_0 & \cdots & E^k_{n_k-1}
\end{array}\right]$$

Fig. 5. Group structure of polyphase matrix of a uniform FB derived from a nonuniform one.

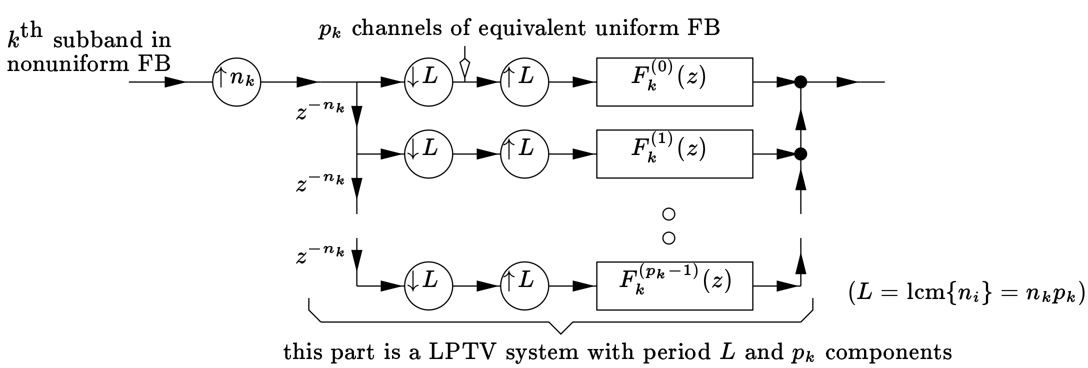

Fig. 6. Using LPTV systems in place of LTI synthesis filters in a nonuniform FB.