FILTER DESIGN FOR CWT COMPUTATION USING THE SHENSA ALGORITHM

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ABSTRACT

Direct computation of CWT using FFT requires $O(Nlog_2N)$ operations per scale, where N is the data length. The Shensa algorithm is a fast algorithm to compute CWT that uses only $O(N)$ operations per scale. The application of the algorithm requires the design of a bandpass and a lowpass filter for a given mother wavelet function. Previous design method involves multi-dimensional numerical search and is computationally intensive. This paper proposes an iterative method to design the optimum filters. It computes in each iteration least-squares solutions only and does not need numerical search. The proposed filter design method is corroborated by simulations.

1. INTRODUCTION

Wavelet transform is a powerful technique to analyse non-stationary signals. It has found application in many areas including image compression, computer vision, signal analysis, sonar and radar, and many others [1, 2, 3, 4]. The continuous wavelet transform (CWT) of a signal $s(t)$ is defined as [4]

$$CWT(a,\tau) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} s(t)\psi^* \left(\frac{t-\tau}{a}\right) dt \qquad (1)$$

where $\psi(t)$ is a mother wavelet function, a and τ are the scale and translation parameters, and the superscript $*$ denotes complex conjugate. The signal $s(t)$ can be reconstructed from the transform coefficients via [4]

$$s(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} CWT(a,\tau) \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-\tau}{a}\right) \frac{1}{a^2} da\, d\tau \qquad (2)$$

provided that the mother wavelet satisfies the adimissibility condition

$$C_\psi = \int_{-\infty}^{\infty} \frac{\Psi(f)}{|f|} df \ < \ \infty \, . \qquad (3)$$

The CWT is often evaluated at dyadic scale $a = 2^j$ and integer translation $\tau = n$. If $s(t)$ is a bandlimited signal, it can be expressed as a weighted sum of its samples $s(n)$. Hence direct computation of CWT using FFT requires $O(N\, log_2 N)$ operations per scale, where N is the data length. This is expensive in practice.

The Shensa algorithm is a fast algorithm that computes dyadic scale CWT at a cost of only $O(N)$ operations per scale. The algorithm is defined by a bandpass filter $\mathbf{h}$ and a lowpass filter $\mathbf{g}$. The mother wavelet function is fixed once the filters are chosen.

Many practical applications such as EEG signal analysis require the computation of CWT using some mother wavelet function that possesses certain properties of interest. In such a case, the Shensa algorithm needs to design $\mathbf{h}$ and $\mathbf{g}$ for a given mother wavelet function $\psi(t)$. Ho *et al.* [5] have shown that the optimum bandpass filter $\mathbf{h}$ can be found by least-squares (LS) minimization. The design of the optimum lowpass filter $\mathbf{g}$, however, requires multi-dimensional numerical search and is computationally expensive. This paper proposes an iterative method to design the two filters. The method involves LS solutions only and does not require numerical search.

The paper is organized as follows. Section 2 is a brief review of the Shensa algorithm [6] and the previous filter design method [5]. Section 3 presents the new filter design method. Section 4 contains the simulation results to corroborate the proposed technique, and Section 5 is the conclusions.

2. THE SHENSA ALGORITHM AND FILTER DESIGN

Figure 1 is the block diagram representation of the Shensa algorithm [6]. The filters $\mathbf{q}$, $\mathbf{h}$ and $\mathbf{g}$ are the initialization filter, bandpass filter and lowpass filter respectively. The input signal in discrete form first passes through the initialization filter $\mathbf{q}$, and then stages of bandpass and lowpass filters. The bandpass filter output at stage j is the approximation of decimated CWT samples $CWT(2^j, 2^{j-1}n)$. The lowpass filter output passes onto the next stage for computing CWT

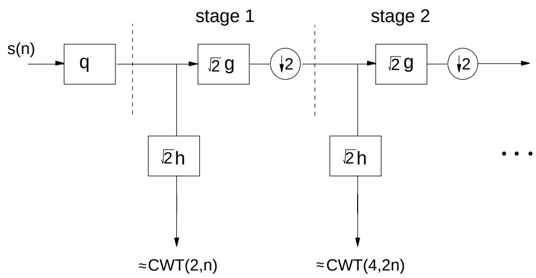

Figure 1: The block diagram of Shansa algorithm to compute decimated CWT samples.

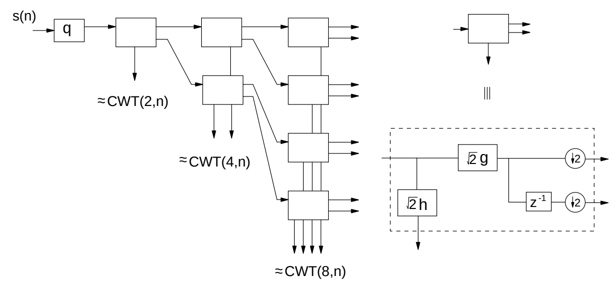

Figure 2: The block diagram of Shensa algorithm to compute undecimated CWT samples.

at the next dyadic scale. The algorithm can be extended to compute undecimated CWT samples. The implementation block diagram in this case is shown in figure 2 [6]. It can be shown that the complexity to compute N undecimated CWT samples using the Shensa algorithm is $O(N)$ operations per scale.

The bandpass filter in the algorithm is related to the wavelet function $\psi(t)$ by

$$\psi(t) = 2 \sum_{k=-L_h}^{L_h} h^*(k)\, \phi(2t+k) \tag{4}$$

where $\phi(t)$ is the scaling function defined by the lowpass filter:

$$\phi(t) = 2 \sum_{k=-L_g}^{L_g} g^*(k)\, \phi(2t+k) \tag{5}$$

and $2L_h + 1$ and $2L_g + 1$ are the lengths of $\mathbf{h}$ and $\mathbf{g}$ respectively. Equations (4) and (5) are called the two-scale equations for the wavelet and the scaling functions. The impulse responses of the lowpass filter are the samples of lowpassed scaling function [7]

$$q(k) = \int \phi^*(t-k) sinc(t)\, dt \tag{6}$$

where $sinc(t) = sin(\pi t)/(\pi t)$. Once the bandpass and lowpass filters are chosen, $\psi(t)$ will be fixed. Here, we are interested in finding the filter pair when a mother wavelet function $\psi_d(t)$ is given.

Let $w_o(t)$ be the equation error from (4):

$$w_o(t) = \psi_d(t) - 2 \sum_{k=-L_h}^{L_h} h^*(k)\phi(2t+k)\,. \tag{7}$$

It is shown in [5] that the mean-square CWT computation error of the Shensa algorithm at scale 2^j is

$$\Gamma_j = 2^j \int_u \int_v r(2^j(u-v)) w_o^*(u) w_o(v)\, du dv \tag{8}$$

where $r(*)$ is the autocorrelation function of $s(t)$. In frequency domain,

$$\Gamma_j = \int_{-2^{j-1}}^{2^{j-1}} P_s\left(f/2^j\right) |W_o(f)|^2\, df \tag{9}$$

where $P_s(*)$ is the power spectral density of the signal, and $W_o(f)$ is the Fourier transform of $w_o(t)$ in (7):

$$W_o(f) = \Psi_d(f) - H^*(f/2)\Phi(f/2)\,. \tag{10}$$

Define $\mathbf{e}_h = \left[\, e^{-j\pi f L_h}\,,\, e^{-j\pi f (L_h - 1)}\,,\, \ldots\,,\, e^{j\pi f L_h} \,\right]^T$ and $\mathbf{h} = [\, h(L_h)\,,\, h(L_h-1)\,,\, \ldots\,,\, h(-L_h)\,]^T$ so that $H(f/2) = \mathbf{e}_h^T \mathbf{h}$. Let j be the stage such that the mean-square error (MSE) Γ_j is largest. Then the mini-max solution of $\mathbf{h}$ that minimizes (9) is [5]

$$\begin{aligned} \mathbf{h} \;=\; & \left(\int_{-2^{j-1}}^{2^{j-1}} P_s(f/2^j) |\Phi(f/2)|^2 \mathbf{e}_h^* \mathbf{e}_h^T df \right)^{-1} \\ & \left(\int_{-2^{j-1}}^{2^{j-1}} P_s(f/2^j) \Psi_d^*(f) \Phi(f/2) \mathbf{e}_h^* df \right). \end{aligned} \tag{11}$$

Putting (11) into (9) yields the minimum MSE:

$$\begin{aligned} \Gamma_{j,min} \;=\; & \int_{-2^{j-1}}^{2^{j-1}} P_s(f/2^j) \Psi_d(f) \\ & \left\{ \Psi_d^*(f) - \Phi^*(f/2) \mathbf{e}_h^T \mathbf{h} \right\} df\,. \end{aligned} \tag{12}$$

Note that both $\mathbf{h}$ and Φ are dependent on $\mathbf{g}$ (c.f. (11) and (5)). The optimum lowpass filter $\mathbf{g}$ is found by multi-dimensional numerical search to further minimize $\Gamma_{j,min}$.

3. NEW FILTER DESIGN METHOD

The previous filter design method is very computationally intensive because of the multi-dimensional search for $\mathbf{g}$. We shall propose a more computationally efficient method for the filter design.

The Fourier transform of (5) is

$$\Phi(f) = G^*(f/2)\Phi(f/2) \tag{13}$$

where

$$G(f) = \sum g(k) e^{-j2\pi f k} \tag{14}$$

is the discrete Fourier transform of the lowpass filter $\mathbf{g}$. Hence iterating (13) gives

$$\Phi(f) = \prod_{k=1}^{\infty} G^*(f/2^k)\ . \tag{15}$$

Substituting (13) into (10) forms

$$W_o(f) = \Psi_d(f) - H^*(f/2)\Phi(f/4)G^*(f/4) \tag{16}$$

so that (9) can be expressed as

$$\Gamma_j = \int_{-2^{j-1}}^{2^{j-1}} P_s\left(f/2^j\right) \left|\Psi_d(f) - H^*(f/2)\Phi(f/4)G^*(f/4)\right|^2 df. \tag{17}$$

Let $\mathbf{e}_g = \left[e^{-j\pi f L_g/2}, e^{-j\pi f(L_g-1)/2}, \ldots, e^{j\pi f L_g/2}\right]^T$ and $\mathbf{g} = \left[g(L_g), g(L_g-1), \ldots, g(-L_g)\right]^T$. Then $G(f/4) = \mathbf{e}_g^T\mathbf{g}$. The lowpass filter coefficients must fulfil certain constraints. In order to guarantee convergence of (15), the sum of the elements of $\mathbf{g}$ must be unity. In addition, imposing a zero of $\mathbf{g}$ at $f = 0.5$ will improve the smoothness in CWT coefficients. Some other contraints on $\mathbf{g}$ may be needed, depending on applications. Denote, in general, the set of linear constraints on the lowpass filter as $\mathbf{Ag} = \mathbf{b}$, where $\mathbf{A}$ is a matrix and $\mathbf{b}$ is a vector. Now suppose $H(f/2)$ is known, the constrained LS solution of $\mathbf{g}$ that minimizes (17) is

$$\mathbf{g} = \tilde{\mathbf{g}} + \mathbf{FA}^\dagger(\mathbf{AFA}^\dagger)^{-1}(\mathbf{b} - \mathbf{A}\tilde{\mathbf{g}}) \tag{18}$$

where

$$\mathbf{F} = \left(\int_{-2^{j-1}}^{2^{j-1}} P_s(f/2^j) \left|H^*(f/2)\Phi(f/4)\right|^2 \mathbf{e}_g^*\mathbf{e}_g^T df\right)^{-1} \tag{19}$$

and

$$\tilde{\mathbf{g}} = \mathbf{F}\left(\int_{-2^{j-1}}^{2^{j-1}} P_s(f/2^j)H^*(f/2)\Phi(f/4)\mathbf{e}_g^*\Psi_d^*(f)df\right) \tag{20}$$

is the unconstrained solution.

The new filter design method is described as follows:

Initialization: select an initial $\mathbf{g}$

Repeat

1. compute $\Phi(f)$ from (15)
2. design $\mathbf{h}$ using (11)
3. design $\mathbf{g}$ using (18)-(20)

until convergence.

The new design method is more computationally efficient than the previous technique. It involves only LS solutions and does not require multi-dimensional numerical search. At present there is no proof regarding global convergence of the method. However, in all the simulation trials the algorithm converged to the global minimum.

4. SIMULATIONS

The mother wavelet functions used in the simulation study were the Morlet wavelet:

$$\begin{aligned} \psi(t) &= e^{-t^2/2\sigma^2} e^{-j2\pi f_o t} \\ \Psi(f) &= \sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2(f-f_o)^2} \end{aligned} \tag{21}$$

and the Second Derivative of a Gaussian wavelet:

$$\begin{aligned} \psi(t) &= (1-t^2)e^{-t^2/2} \\ \Psi(f) &= (2\pi)^{5/2} f^2 e^{-2\pi^2 f^2}\ , \end{aligned} \tag{22}$$

where $f_o = 0.6$ and $\sigma = 0.2$.

The length of $\mathbf{h}$ was 33 and that of $\mathbf{g}$ was 5. The initial $\mathbf{g}$ was set to $[1\ 4\ 6\ 4\ 1]^T/16$, which corresponds to a B-spline scaling function. The lowpass filter was constrained to have a unity gain at zero frequency and a zero gain at frequency=0.5. Furthermore, it was also constrained to be real and symmetric. The number of stages for CWT computation was 8.

Figure 3 shows the decrease in MSE as the number of iteration increases. The mother wavelet function was Morlet. The input signal was a unit impulse, and the maximum MSE occurred in the last stage, and j was set to 8 in computing the LS solutions. The method converges in about 25 iterations and reaches a minimum MSE at -35dB. The dotted line is the minimum MSE found by numerical search. It is clear that the proposed method converges to the global minimum.

Figure 4 is the case when the mother wavelet function was Second Derivative of a Gaussian. The proposed method reaches the global minimum MSE of -62dB in about 25 iterations.

Figure 5 is the result when the input was a correlated AR(1) process given by $s(k) = 0.85s(k-1) + \rho(k)$, where ρ is a white random process whose power is such that the power of $s(k)$ is unity. The mother wavelet function was Morlet. The largest MSE occurred in the last stage and j was set to 8 in computing the LS solutions. Again, the proposed method converges at about 25 iterations and reaches a minimum MSE of -24dB. The case for the Second Derivative of Gaussian wavelet function is given in figure 6. The observations are similar with the Morlet wavelet case.

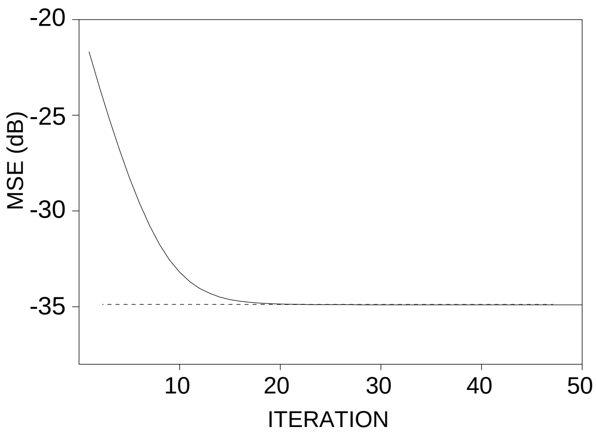

Figure 3: Convergence curve of the proposed method, unit sample input and Morlet wavelet.

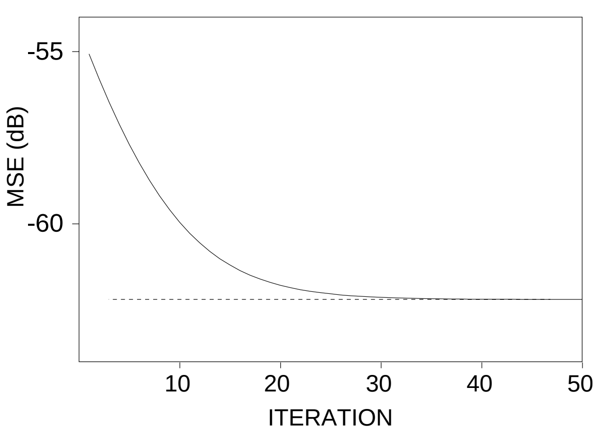

Figure 4: Convergence curve of the proposed method, unit sample input and Second Derivative of Gaussian wavelet.

5. CONCLUSIONS

We have proposed a new method to design the bandpass and lowpass filters in the Shensa algorithm for the computation of CWT when a mother wavelet function is given. The method is iterative and requires only LS solution computations. The convergence and validity of the method were confirmed by simulations.

6. REFERENCES

[1] O. Rioul and M. Vetterli, ”Wavelets and signal processing,” *IEEE Signal Processing Mag.*, vol. 8, no. 4, pp. 11-38, Oct. 1991.

[2] M. Antonini, M. Barlaud, P. Mathieu and I. Daubechies, ”Image coding using vector quantization in the wavelet transform domain,” *in Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, Albuquerque, NM 1990, pp. 2297-2300.

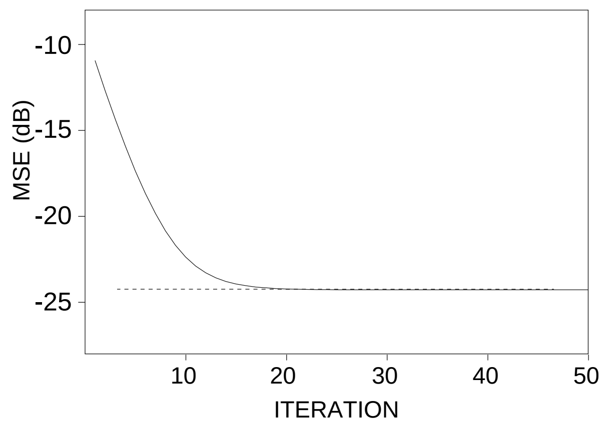

Figure 5: Convergence curve of the proposed method, correlated input and Morlet wavelet.

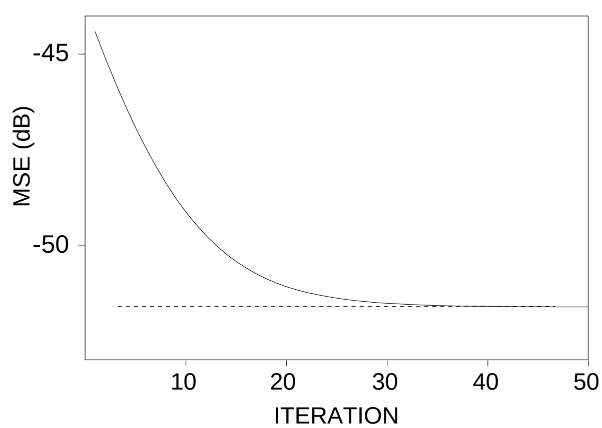

Figure 6: Convergence curve of the proposed method, unit sample input and Second Derivative of Gaussian wavelet.

[3] S. G. Mallat, ”Multifrequency channel decompositions of images and wavelet models,” *IEEE Trans. Acoust., Speech, Signal Proccessing*, vol. 37, pp. 2091-2110, Dec. 1989.

[4] Y. T. Chan, *Wavelet Basics*. Kluwer Academic Publishers, Boston, 1995.

[5] K. C. Ho and Y. T. Chan, “Filter design and comparison for two fast CWT algorithms,” submitted to *IEEE Trans. Signal Processing*.

[6] M. J. Shensa, “The discrete wavelet transform: wedding the à trous and Mallat algorithms,” *IEEE Trans. Signal Processing*, vol. 40, pp. 2464-2682, Oct. 1992.

[7] X. G. Xia, C. C. J. Kuo and Z. Zhang, “Wavelet coefficient computation with optimal prefiltering,” *IEEE Trans. Signal Processing*, vol. 42, pp. 2191-2197, Aug. 1994.