WARPED LINEAR TIME INVARIANT SYSTEMS AND THEIR APPLICATION IN AUDIO SIGNAL PROCESSING

John Garas and Piet C.W. Sommen

Eindhoven University of Technology
Dept. Electrical Engineering, Building Eh 6.34
P.O.Box 513, 5600 MB Eindhoven, The Netherlands
e-mail: `j.garas@ele.tue.nl`
URL: `http://www.ses.ele.tue.nl/~garas/`

ABSTRACT

The main goal behind coordinate transformation (warping) of a Linear Time Invariant (LTI) system is to represent its signals in terms of new basis functions that better suit the application in hand. Unitary operators simplify the analysis considerably; therefore, they are used to derive the relations between variables in the original and warped domains. These relations show that an LTI system can be warped by processing its input signals with a unitary warping transformation. An efficient implementation of this warping transformation, that is based on a nonuniform sampling theorem, is given; which allows applying the warping principle in real-time applications. As an example of exploiting this technique, it is shown that sampling an audio signal at exponentially spaced moments changes the underlying coordinates of its signal processing system to suit those of the human auditory system.

1. INTRODUCTION

The concept of coordinate transformation, better known as warping, has been used extensively in many applications such as transform coding, adaptive filters and dynamic time-warping of speech signals. Recently, Baraniuk and Jones [1] have extended its application domain to joint signal representations of arbitrary variables including time-frequency and time-scale distributions. The main goal behind coordinate transformation is to represent signals and systems in terms of new basis functions that better suit the application in hand. A good example may be found in audio signal processing. It is known that the human auditory system performs nonuniform spectral analysis on sound waves collected by the pinnae. Therefore, audio applications that interact with the auditory system should have similar internal representations of signals. It is shown in Section 5 that warping the coordinates of the signal processing system to match those of the human auditory system is sufficient to produce similar nonuniform analysis. But first, the theory behind coordinate warping is presented. In Section 2, the required basics of unitary operators are introduced. In Section 3, the unitary equivalence principle is used to derive the relations between variables and operators in time and frequency domains and their warped counterparts. An efficient implementation of the coordinate warping transformation using a nonuniform sampling theorem is derived in Section 4. Finally, Section 6 sums up the important messages conveyed by the paper.

Throughout the paper, lower case letters are used to represent signals in time domain while upper case letters represent signals in frequency domain. The symbol $\mathbb{F}$ is used to denote the Fourier operator. All signals are considered to be elements of the Hilbert space of square-integrable functions $L^2(\mathbb{R})$, which has inner product $\langle s,h\rangle = \int_{\mathbb{R}} s(\tau)h^*(\tau)d\tau$ for $s,h \in L^2(\mathbb{R})$ and norm $\|h\|^2 = \langle h,h\rangle$, where $(\cdot)^*$ denotes the complex conjugate. Operators on the Hilbert space are expressed using boldface capital letters. The notation $(\mathbf{U}\ s)(x)$ is used to denote processing the signal s by the operator $\mathbf{U}$ and evaluating the result at x.

2. UNITARY OPERATORS

A unitary operator $\mathbf{U}$ is a linear transformation that maps the Hilbert space into itself. Unitary operators preserve energy and inner products; $\|\mathbf{U}s\|^2 = \|s\|^2$ and $\langle \mathbf{U}s, \mathbf{U}h\rangle = \langle s,h\rangle$. As a consequence, a unitary operator maps a set of orthonormal bases in $L^2(\mathbb{R})$ into another set of orthonormal bases in $L^2(\mathbb{R})$. The class of unitary operators that is of interest to us in this paper is the class of unitary coordinate transformations (axis warping). A subclass of such unitary coordinate transformations on $L^2(\mathbb{R})$ is given by [1]

$$(\mathbf{U}s)(x) = \sqrt{|\dot{\gamma}(x)|}\ s(\gamma(x)) \tag{1}$$

where $\gamma(x)$ is a smooth, monotonic and one-to-one function, and $\dot{\gamma}(x) = d\gamma(x)/dx$. The weighting term $(\sqrt{|\ \dot{\gamma}(x)\ |})$

preserves the signal energy, and therefore ensures that the transformation is unitary. In Section 4, we will show that the transformation (1) can be implemented efficiently using nonuniform sampling of the signal s. To reach this point, we need to define two other classes of unitary operators. The first is the class of parametrized unitary operators representing physical quantities such as time and frequency. In this representation, a variable a is associated with the operator $\mathbf{A}_a$ parametrized by a. The second is the class of unitary signal transformations acting as density functions for physical quantities, such as the Fourier transformation. The latter class can be obtained by projecting a signal s onto the eigenfunctions of the operator $\mathbf{A}_a$. The eigenfunctions and eigenvalues of the parameterized unitary operator $\mathbf{A}_a$ can be found by solving the eigenequation $(\mathbf{A}_a\ \mathbf{e}_k^{\mathbf{A}})(x) = \lambda_{a,k}^{\mathbf{A}}\ \mathbf{e}_k^{\mathbf{A}}(x)$. This yields the eigenfunctions $\{\mathbf{e}_k^{\mathbf{A}}(x); k \in \mathbb{R}\}$ and the eigenvalues $\{\lambda_{a,k}^{\mathbf{A}}; k \in \mathbb{R}\}_a$. The expansion of a signal s onto these eigenfunctions (referred to as the $\mathbf{A}$-Fourier transform $\mathbb{F}_{\mathbf{A}}$ in [1]) is given by

$$S(k) = (\mathbb{F}_{\mathbf{A}}\ s)(k) = \langle s, \mathbf{e}_k^{\mathbf{A}} \rangle = \int_{\mathbb{R}} s(x)\ \mathbf{e}_k^{\mathbf{A}^*}\ dx \qquad (2)$$

and the inverse transformation is given by

$$s(x) = (\mathbb{F}_{\mathbf{A}}^{-1}\ S)(x) = \langle S, \mathbf{e}_k^{\mathbf{A}^*} \rangle = \int_{\mathbb{R}} S(k)\ \mathbf{e}_k^{\mathbf{A}}\ dk \qquad (3)$$

The time-shift operator corresponding to the time variable and the frequency-shift operator corresponding to the frequency variable together with their corresponding eigenfunctions are summarized in Table 1. The transformation $\mathbb{F}_{\mathbf{T}} = \langle s, \mathbf{e}_k^{\mathbf{T}} \rangle$ can be identified as the usual Fourier transform $\mathbb{F}$ which is invariant to time shifts up to a phase factor $|(\mathbb{F}\ \mathbf{T}_\mu s)(k)| = |(\mathbb{F}\ s)(k)|$, covariant to frequency shifts $(\mathbb{F}\ \mathbf{F}_\nu s)(k) = (\mathbb{F}\ s)(k+\nu)$ and measures the frequency contents in the signal $s(x)$. The transformation $\mathbb{F}_{\mathbf{F}} = \langle s, \mathbf{e}_k^{\mathbf{F}} \rangle$ corresponds to the mirror transform $(\mathbb{F}_{\mathbf{F}}\ s)(x) = s(-x)$ that is invariant to frequency shifts, covariant to time shifts and measures time contents in the signal $s(x)$.

Operator	Definition in t-domain	Eigenfunctions
Time shift	$(\mathbf{T}_\mu\ s)(x) \equiv s(x-\mu)$	$\mathbf{e}_k^{\mathbf{T}}(x) = e^{j2\pi kx}$
Freq. shift	$(\mathbf{F}_\nu\ s)(x) \equiv e^{j2\pi\nu x}\ s(x)$	$\mathbf{e}_k^{\mathbf{F}}(x) = \delta(x+k)$

Table 1: Time and frequency operators and their eigenfunctions.

3. UNITARY EQUIVALENCE AS A COORDINATE TRANSFORMATION

It is a well known property of the Fourier transform that a time signal $s(t)$ can be modulated (frequency shifted) by either multiplying it by $e^{j2\pi\alpha t}$ or by translating its Fourier transform: $\mathbf{F}_\alpha = \mathbb{F}^{-1}\ \mathbf{T}_\alpha\ \mathbb{F}$. Similarly $\mathbf{T}_\beta = \mathbb{F}^{-1}\ \mathbf{F}_\beta\ \mathbb{F}$

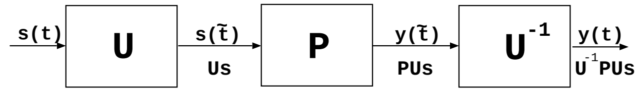

Figure 1: Warped linear time invariant system.

expresses time shift, which is equivalent to multiplying $(\mathbb{F}\ s)$ by the phase factor $e^{j2\pi\beta k}$. This principle of operator equivalence modulo a unitary transformation can be generalized to any unitary operator; which leads to the *unitary equivalence* principle [1]. This principle states that two operators $\widetilde{\mathbf{A}}$ and $\mathbf{A}$ are unitary equivalent if $\widetilde{\mathbf{A}} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}$, with $\mathbf{U}$ a unitary transformation. Solving the eigenequation of $\widetilde{\mathbf{A}}$ yields the new eigenfunctions $\mathbf{e}_k^{\widetilde{\mathbf{A}}} = \mathbf{U}^{-1}\mathbf{e}_k^{\mathbf{A}}$, while the two operators have the same eigenvalues. And the $\widetilde{\mathbf{A}}$-Fourier transform is given by $\mathbb{F}_{\widetilde{\mathbf{A}}} = \mathbb{F}_{\mathbf{A}}\ \mathbf{U}$. Since $\mathbf{U}$ is unitary, it preserves the inner product on $L^2(\mathbb{R})$ such that $\langle \widetilde{\mathbf{A}}, \widetilde{\mathbf{B}} \rangle = \langle \mathbf{A}, \mathbf{B} \rangle$ for any two operators $\mathbf{A}$ and $\mathbf{B}$.

Application of the unitary equivalence principle to the time and frequency operators $\mathbf{T}$ and $\mathbf{F}$ results in two new operators $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{F}}$ that correspond to two new variables $\tilde{t}$ and $\tilde{f}$, respectively:

$$\widetilde{\mathbf{T}}_\mu = \mathbf{U}^{-1}\mathbf{T}_\mu\mathbf{U} \qquad \widetilde{\mathbf{F}}_\nu = \mathbf{U}^{-1}\mathbf{F}_\nu\mathbf{U} \qquad (4)$$

When the unitary transformation is of the warping type given by (1), we will refer to $\tilde{t}$ and $\tilde{f}$ as warped-time and warped-frequency, respectively. From the above, the $\widetilde{\mathbf{T}}$-Fourier transform $\mathbb{F}_{\widetilde{\mathbf{T}}}$ is given by $\mathbb{F}_{\widetilde{\mathbf{T}}} = \mathbb{F}_{\mathbf{T}}\ \mathbf{U} = \mathbb{F}\ \mathbf{U}$ and can be shown to be invariant to $\widetilde{\mathbf{T}}$ and covariant to $\widetilde{\mathbf{F}}$ (while $\mathbb{F}$ is invariant to $\mathbf{T}$ and covariant to $\mathbf{F}$). Similarly, the $\widetilde{\mathbf{F}}$-Fourier transform is given by $\mathbb{F}_{\widetilde{\mathbf{F}}} = \mathbb{F}\mathbb{F}\ \mathbf{U} = \mathbf{M}\mathbf{U}$; where $\mathbf{M}$ is the mirror operator $(\mathbf{M}s)(t) = s(-t)$. Similarly, $\mathbb{F}_{\widetilde{\mathbf{F}}}$ can be shown to be invariant to $\widetilde{\mathbf{F}}$ and covariant to $\widetilde{\mathbf{T}}$.

The underlying coordinates of an LTI system $\mathbf{P}$ can be changed by preprocessing its input signal with the unitary transformation $\mathbf{U}$ as shown in Fig. 1. This preprocessing transforms the input signal $s \longmapsto \mathbf{U}s$ and maps the operators $\mathbf{T} \longmapsto \widetilde{\mathbf{T}}$ and $\mathbf{F} \longmapsto \widetilde{\mathbf{F}}$; corresponding to the warped-time and warped-frequency variables as discussed above. The system output $y(\tilde{t}) = (\mathbf{P}\mathbf{U}s)(t)$ can be considered as the response of the warped-LTI (WLTI) system given by $\mathbf{P}\mathbf{U}$. Since an LTI is characterized by its covariance to time shifts, the original system $\mathbf{P}$ is covariant to $\mathbf{T}$: $(\mathbf{P}\mathbf{T}_\mu s)(t) = (\mathbf{P}s)(t-\mu)$. The WLTI system $\mathbf{P}\mathbf{U}$ however, is covariant by translation not to $\mathbf{T}$ but to $\widetilde{\mathbf{T}}$: $(\mathbf{P}\mathbf{U}\widetilde{\mathbf{T}}_\mu s)(\tilde{t}) = (\mathbf{P}\mathbf{U}s)(\tilde{t}-\mu)$. Therefore, by carefully choosing the unitary preprocessing operator $\mathbf{U}$, it is possible to "rotate" the system coordinates to better suit prespecified requirements on the resulting WLTI system. In many cases, the output of the LTI system has to be presented with respect to the original coordinates, and the inverse warping $\mathbf{U}^{-1}$ in Fig. 1 is needed.

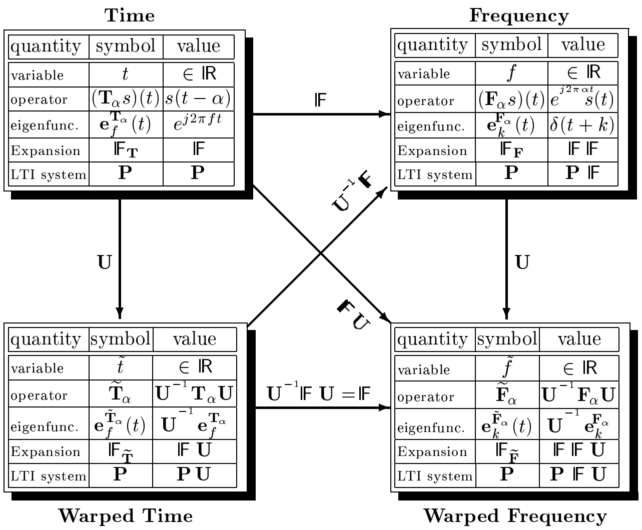

Figure 2: Relations between time domain, frequency domain and their warped counterparts.

The relations between time domain, frequency domain and their warped counterparts are summarized in Fig. 2. In this figure, an arrow with an operator $\mathbf{X}$ on top indicates the required transformation to move from one domain to the other in the arrow direction. The transformation in the opposite direction is performed by $\mathbf{X}^{-1}$. Note that the unitary transformation $\mathbf{U}$ is defined to be the same for time and frequency signals, therefore $\mathbb{F}\ \mathbf{U} = \mathbf{U}\ \mathbb{F}$.

4. NONUNIFORM SAMPLING AS COORDINATE TRANSFORMATION

In real-time digital signal processing (DSP) systems, processing speed is of high importance. Therefore, an efficient implementation of the unitary warping transformation (1) is required to put the warping principle into application. In general, calculating the weighting function $\sqrt{|\dot{\gamma}(t)|}$ forms no difficulties since γ must be chosen to be a smooth, monotonic and one-to-one function. In most cases, this can be done off-line and the result stored in a lookup table to be accessed during real-time processing. Therefore, in the rest of this section, we will consider only the second factor of the right hand side of (1).

For most real-time DSP systems, input and output signals are continuous-time signals. Analog-to-digital converters (ADC) are used to sample the input signals, while digital-to-analog converters (DAC) are used to transform the output signals back to their continuous-time form. Considering that the sampling process is performed after warping, the warped signal $(\mathbf{U}s)[nT]$ (see Fig. 1) can be expressed as

$$(\mathbf{U}s)[nT] = \sqrt{|\dot{\gamma}[nT]|}\ s[\gamma(nT)] \qquad (5)$$

where T is the sampling period that is considered to comply with the classical sampling theorem. Equation (5) can be interpreted as sampling the signal $s(t)$ at new sampling moments given by $t_n = \gamma(nT)$, thus producing nonuniformly sampled signal. The following theorem shows that this nonuniformly sampled signal is guaranteed to be free from aliasing.

Clark's nonuniform sampling theorem [2]:
Let a function $f(t)$ be sampled at the sampling moments $t = t_n$, where t_n is not necessarily a sequence of uniformly spaced numbers. If a one-to-one continuous mapping $\xi(t)$ exists such that $nT = \xi(t_n)$, and if $g(\tau) = f(\xi^{-1}(\tau))$ is band limited to $\omega_o = \pi/T$, then the function $f(t)$ can be reconstructed exactly from its samples using the interpolation formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n)\, \frac{\sin\left[\frac{\pi}{T}\left(\gamma(t)\ -\ nT\right)\right]}{\left[\frac{\pi}{T}\left(\gamma(t)\ -\ nT\right)\right]} \qquad (6)$$

In the case of the unitary warping transformation (5), since we started off with a warping function γ that is smooth, monotonic and one-to-one, the mapping $\xi^{-1}(\cdot) = \gamma(\cdot)$ is guaranteed to exist. Moreover, since $\mathbf{U}$ is a unitary transformation, its inverse exists and the inverse warping function $\gamma^{-1}(\cdot) = \xi(\cdot)$ exists, which proves that the system is free from aliasing. Therefore, sampling s at the moments $t_n = \gamma(nT)$ and multiplying the result by the weighting factor $\sqrt{|\dot{\gamma}[nT]|}$ is a valid implementation of the unitary warping transformation (1). The total complexity required for warping a signal of time interval NT seconds is reduced to N real multiplications that are required for the weighting factor $\sqrt{|\dot{\gamma}(t)|}$ in (5). In cases when input signals are already in digital form, the nonuniform sampling can still be used by using a hardware resampler as described in [3].

5. WARPED AUDIO SIGNAL PROCESSING SYSTEMS

In this section, the auditory filter bank problem, mentioned in the introduction, is used as an example to present the design procedure of WLTI systems. This example also expresses the potential improvement of conventional audio systems when combined with a preprocessing warping stage.

Consider an LTI system $\mathbf{P}$ that functions as an audio spectral analyzer. The input signal is first sampled uniformly with sampling period T seconds according to the classical sampling theorem. Blocks of N samples are collected, an N-point FFT is calculated and the result is presented. The output of this system is shown in Fig. 3-b for $N = 64$, $T = 1/300$ seconds and the input signal (shown in Fig. 3-a) is given by $s(t) = \sum_{n=1}^{4} \cos(2\pi \frac{0.1\,n}{T} t)$

Since the FFT algorithm calculates equally spaced samples of the signal's Fourier transform, this simple analyzer can be considered as a constant-bandwidth filter bank. On the contrary, the human auditory system performs constant-Q "percentage-bandwidth" analysis in most of the audio frequency range. Therefore, the system $\mathbf{P}$ does not present a true estimate of the signal components as seen by the human auditory system, and can not accurately resolve known phenomena such as frequency masking. In order to take these perception issues into account, we need to convert the constant-bandwidth filter bank, performed by the above system, to a constant-Q filter bank, preferably with as little extra computation as possible. Several methods exist that can perform the task, a survey of which can be found in [3]. Unfortunately, all of the existing techniques require a great deal of computation complexity. The warping principle proves to be very efficient in this case: warping the frequency coordinate f of the above system to $\gamma(f) = f_0 \cdot a^f$ and performing the same FFT on the warped signal is sufficient. In the above warping function, the variable $a \in \mathbb{R},\ a > 1$ controls the bin spacing and bandwidth on the warped frequency coordinate. $f_0 \in \mathbb{R}_+$ is a reference frequency taken to be the smallest frequency of interest. This maps the frequency sampling points of the FFT from $f_k = kF$ to $f_k = f_0\,(a^{kF})$; where k is the frequency index and $F = 1/NT$ is the bin spacing of the ordinary FFT. From Fig. 2, this is equivalent to considering $\mathbf{P} = \mathbf{I}$ and performing a transformation $\mathbb{F}\,\mathbf{U}$ from time to warped frequency. This allows us to use the nonuniform sampling method to implement the warping efficiently, which is not possible if we consider a transformation from frequency to warped frequency. Using a similar time-warping function and substituting in (5) gives the required unitary warping transformation

$$(\mathbf{U}s)[nT] = \sqrt{|a^{nT}\ \log(a)|}\ s[t_0\ a^{nT}] \tag{7}$$

where $t_0 \in \mathbb{R}_+$ is an arbitrary time reference on t and $a \in \mathbb{R}$, $a > 1$. This transformation maps functions on $L^2(\mathbb{R})$ onto functions on $L^2(\mathbb{R}_+)$. From Fig. 2, it is readily evaluated that $(\widetilde{\mathbf{T}}_\alpha s)(t) = a^{-\alpha/2} s(a^{-\alpha} t)$ which is a dilation operator, $(\widetilde{\mathbf{F}}_\alpha s)(t) = e^{j2\pi\alpha \log_a(t)} s(t)$ corresponding to logarithmic modulation. The Fourier transform in this case is $(\mathbb{F}_{\widetilde{\mathbf{T}}}\ S)(f) = c\int_0^\infty \frac{s(t)}{\sqrt{t}} e^{j2\pi f \log_a(t)} dt$; where $c = \sqrt{\log(a)}$ which is the Mellin transform known for its constant-Q analysis.

Since the FFT considers the input signal to be periodic with a period N, we only need to warp this time period of N samples. The output of the WLTI spectral analyzer is shown in Fig. 3-d, and the nonuniform sampled input signal in Fig. 3-c for $a = e$ and $t_0 = 1$. It is clear from this result that the WLTI system indeed performs spectral decomposition similar to the auditory system; it produces higher resolution at low frequencies and lower resolution at high frequencies.

Finally, it is worth mentioning that although N extra multiplications are consumed every block to realize the warping, it is compensated by using a more efficient sampling mechanism. Note from Fig. 3 that the same number of samples $N = 64$ are collected in longer time in case (c) than in case (a) which leads to less number of blocks per unit time.

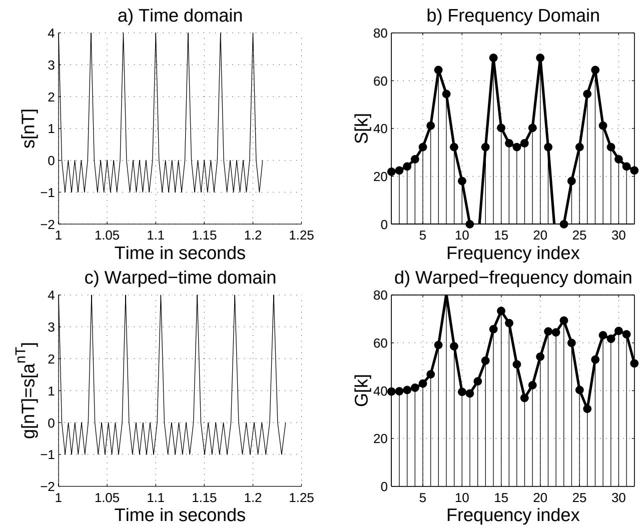

Figure 3: Spectral analysis in time and warped-time domains.

6. CONCLUSIONS

It is shown that a Linear Time Invariant (LTI) system can be readily warped by sampling its input signals nonuniformly. Nonuniform sampling not only implements the warping unitary transformation efficiently, but also leads to a more efficient sampling mechanism for many warping functions. Having such an efficient implementation allows utilizing the powerful tool of coordinate transformation in real-time signal processing systems. This technique offers audio applications a simple way of adjusting their performance to take the human auditory system behaviour into account.

7. REFERENCES

[1] R. G. Baraniuk, D. L. Jones: *Unitary Equivalence: A New Twist on Signal Processing*, IEEE Trans. Signal. Process., Vol. 43, No. 11, pp. 2269-2282, Oct. 1995.

[2] J. Clark, M. Palmer, and P. Lawrence: *A Transformation Method for the Reconstruction of Functions from Nonuniformly Spaced Samples*, IEEE Trans. Vol. ASSP-33, No. 4, pp. 1151-1165, Oct. 1985.

[3] J. Garas and P.C.W. Sommen: *Improving Virtual Sound Source Robustness Using Multiresolution Spectral Analysis and Synthesis*, AES 105th Convention, preprint no. 4824, San Francisco, Sept. 1998.