A DATA-DRIVEN SCHEME FOR THE APPROXIMATED COMPUTING OF ALIAS-FREE GENERALIZED DISCRETE TIME-FREQUENCY DISTRIBUTIONS

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ABSTRACT

Time-Frequency Distribution (TFD) based on Cohen's class has significant potential for the analysis of a number of non-stationary signals. One of the discrete formulations is the recently introduced Alias-Free Generalized Discrete-Time TFD (AF-GDTFD). The spectral decomposition of the kernel allows the computation of AF-GDTFD as a weighted sum of spectrograms. The partial sum has been shown to offer a vehicle to trade-off between exactness and computational load. This paper proposes a scheme which exploits local approximations by adapting dynamically the accuracy of spectrograms to the eigenvalue magnitudes. The approach employs the wavelet packet transform followed by a block-recursive Fourier transform and a compensation network. Adaptive selection of subbands for further processing reduces substantially the computational cost while still preserving an acceptable quality. The approach is attractive in terms of VLSI aspects due to the modular structure, local connections and stream processing.

1. INTRODUCTION

Spectral analysis of the time-varying frequency content of signals is an important task in many DSP applications. Classical method relies on the evaluation of short-time Fourier transform (STFT) since the Fourier method is well understood and efficient implementation techniques built around the Fast Fourier Transform (FFT) are available. The class of bilinear signal transformation defined by Cohen [1], which includes the spectrogram and the Wigner-Ville distribution, provides a richer tool for non-stationary signal analysis. Each of these members is associated with a kernel function which is independent of the signal. These kernels mainly differ in their smoothing behavior in the time-frequency domain and have been successfully applied to a wide range of application in terms of off-line analysis [2]. The main drawback of these Time-Frequency Distributions (TFDs) is their high computational requirement which limits their use in systems where hardware resources (power consumption, area) are costly. Different approaches have been proposed to accelerate the TFD computation, especially for time-frequency representation of long signals. Most of these approaches take the advantages of the symmetric structure of the distribution kernel and autocorrelation matrix of the signals to reduce the computation. Other approaches look for recursive implementation when using some specific running windows. A method proposed by [3] considers the eigen-decomposition of the kernel for the computation of the Alias-Free Generalized Discrete-Time TFD (GDTFD) in [4]. Due to available packages in hardware and software for spectrogram-based evaluation, this computation method provides a smooth transition from the classical and well-known STFT to kernel-based TFD.

The paper is organized as follows: first, we review the idea of AF-GDTFD shortly. Next, the main structure of our data-driven adaptive scheme is presented which is based on the wavelet packet transform (WPT) to divide the input signal into subbands and a block-recursive FFT to perform the required down-sampling in frequency. A detailed complexity discussion is then followed by an example to prove the effectiveness of the scheme.

2. EIGEN-DECOMPOSITION BASED AF-GDTFD

The class of bilinear TFD [1] is a conceptual and theoretical improvement compared to windowed spectral analysis. The processing of discrete samples $x(n)$ requires a discrete version of TFD. One of these formulations is the AF-GDTFD introduced in [4]:

$$TFD_{AF}(n,\omega) = \sum_m R_x^G(n,m)e^{-j\omega m}, \quad (1)$$

$$R_x^G(n,m) = \sum_l \psi(n-l,m)R_x(l,m), \quad (2)$$

where $\psi(n,m)$ is the kernel in the discrete time-lag (autocorrelation) domain and $R_x^G(n,m)$ is defined for all $\{n,m \in \mathbb{Z} : n+\frac{m}{2} \in \mathbb{Z},\ n-\frac{m}{2} \in \mathbb{Z}\}$. $\mathbb{Z}$ is the set of all integers, and $R_x(n,m) = x(n+\frac{m}{2})\ x^*(n-\frac{m}{2})$ is the discrete instantaneous autocorrelation. For a real-valued AF-GDTFD, $\psi(n,m)$ is conjugate symmetric and has finite non-zero support in

$$\{n,m \in \mathbb{Z} : -(L-1) \leq \begin{matrix} n+\frac{m}{2} \\ n-\frac{m}{2} \end{matrix} \leq L-1\}. \quad (3)$$

After a variable transformation on (n,m), the resulting shifted kernel $\tilde{\psi}(n_1,n_2)$ can be represented by an eigen-decomposition

$$\tilde{\psi}(n_1,n_2) = \sum_{k=1}^{2L-1} \lambda_k \mathbf{v}_k \mathbf{v}_k^H, \quad (4)$$

where $\mathbf{v}_k = \{v_k(n)\}_{n=1}^{2L-1}$ [3]. Combining Eq. 1, 2 and 4, TFD_{AF} can be expressed as a finite, weighted sum of discrete-time spectrograms by

$$\mathbf{tfd}_{AF}(n) = \sum_{k=1}^{2L-1} \lambda_k \cdot \mathbf{u}^H \mathbf{I}_L \mathbf{u}, \quad (5)$$

$$\mathbf{u} = (\downarrow 2)\mathbf{F}_{2L-1} \cdot \mathbf{V}_k^* \cdot \mathbf{x}(n), \quad (6)$$

where $\mathbf{tfd}_{AF}(n) = [TFD_{AF}(n,\omega_0)\dots TFD_{AF}(n,\omega_{L-1})]^T$ is a time-varying column vector evaluated at $\omega_l = \{\frac{2\pi l}{L}\}_{l=0}^{L-1}$. $\mathbf{F}_{2L-1}$ is the Fourier matrix of size $2L-1$, $\mathbf{V}_k^*$ is a diagonal matrix with $\mathbf{v}_k^*$ on its diagonal. $\mathbf{x}(n) = [x(n-L+1)\dots x(n)\dots x(n+L-1)]^T$ is the column data vector. $(\downarrow 2)$ specifies the decimation by 2 which removes the odd rows (due to symmetric structure of $R_x^G(n,m)$. $\mathbf{I}_L$ is the indentity matrix of size $L \times L$.

The computation at each time step is $2L-1$ weighted spectrograms of size $2L-1$ with a frequency resolution of $\frac{2\pi}{L}$. A straight-forward scheme would be the padding by one zero in order to take advantage of FFT (from here on $N = 2L$ for short). An approximation of the calculation can be achieved by considering only the partial sum over the $\hat{L}$ largest λ_k in magnitude. However, the number of eigenvalues used to compute $\mathbf{tfd}_{AF}$ provides only limited degree of trade-offs between spectral quality and computational time.

3. AF-GDTFD APPROXIMATED COMPUTING

It is apparent that the accuracy of the spectrogram remains the same, independent of the eigenvalue magnitudes. Besides the global approximation by restricting to $\hat{L}$ significant eigenvalues, we would like to have a scheme which allows arbitrary frequency coverage and frequency resolution for the local approximation. We also would like to do the approximation on an adaptive basis to account for the time-varying nature of real-world signals. The main idea of the approach is that the accuracy of each spectrogram in Eq. 5 can be sacrificed with decreasing magnitude of its associated eigenvalue. The second issue is the application of WPT as a subband preprocessing step, to break down a large size FFT into smaller FFT blocks in parallel and further select subbands to be processed using some significance measures.

3.1. WPT-based Precomputing

The basic block of the wavelet transform is a two-channel filter bank with the analysis filters $\tilde{\mathbf{g}}$ (high-pass) and $\tilde{\mathbf{h}}$ (low-pass) followed by down-sampling. The inverse transform first up-samples and uses two synthesis filters $\mathbf{g}$ (high-pass) and $\mathbf{h}$ (low-pass). For perfect reconstruction, the four filters have to fulfill a set of relations. For details, we refer to e.g. [5]. For the general case, the time-domain analysis matrix $\mathbf{W}_N$ and synthesis matrix $\mathbf{W}_N^{-1}$ of size $N \times N$ is of the double-shift type

$$\mathbf{W}_N = \begin{bmatrix} \tilde{h}_0 & \tilde{h}_1 & & \cdots & & \tilde{h}_{\tilde{J}-1} & 0 & 0 & 0 \\ 0 & 0 & \tilde{h}_0 & \tilde{h}_1 & & \cdots & & \tilde{h}_{\tilde{J}-1} & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ \tilde{h}_2 & \tilde{h}_3 & \cdots & \tilde{h}_{\tilde{J}-1} & & \cdots & 0 & \tilde{h}_0 & \tilde{h}_1 \\ \tilde{g}_0 & \tilde{g}_1 & & \cdots & & \tilde{g}_{J-1} & 0 & 0 & \cdots \\ 0 & 0 & \tilde{g}_0 & \tilde{g}_1 & & \cdots & & \tilde{g}_{J-1} & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ \tilde{g}_2 & \tilde{g}_3 & \cdots & \tilde{g}_{J-1} & & \cdots & 0 & \tilde{g}_0 & \tilde{g}_1 \end{bmatrix} \quad (7)$$

$$\mathbf{W}_N^{-1} = \begin{bmatrix} h_0 & 0 & 0 & h_{J-2} & g_0 & 0 & 0 & g_{\tilde{J}-2} \\ h_1 & 0 & 0 & h_{J-1} & g_1 & 0 & 0 & g_{\tilde{J}-1} \\ \vdots & h_0 & 0 & 0 & \vdots & g_0 & 0 & 0 \\ \vdots & h_1 & 0 & \vdots & \vdots & g_1 & 0 & \vdots \\ h_{J-1} & \vdots & \vdots & h_0 & g_{\tilde{J}-1} & \vdots & \vdots & g_0 \\ \vdots & \vdots & \vdots & h_1 & \vdots & \vdots & \vdots & g_1 \end{bmatrix} \quad (8)$$

According to [6], we give $\mathbf{W}_N$ in a circulant structure to deal with finite sequence. Other methods such as reflections, interpolation, and boundary filters can also be used but it would not affect our argument significantly. We also assume that h_0 and h_1 (or $\tilde{h}_0, \tilde{h}_1$) are not both zero, but the number of the filter taps (J, $\tilde{J}$) could vary. This would also include the biorthogonal case.

Exploiting the fact that $\mathbf{W}_N^{-1}\mathbf{W}_N = d\mathbf{I}_N$ ($d = 1$ in the orthogonal case), the factorization of the Fourier matrix $\mathbf{F}_N$ can be achieved by

$$\begin{aligned} \mathbf{F}_N &= d^{-1} \cdot \mathbf{F}_N \mathbf{W}_N^{-1} \mathbf{W}_N \quad (9) \\ &= d^1 \cdot \begin{bmatrix} \mathbf{A}_{N/2} & \mathbf{B}_{N/2} \\ \mathbf{C}_{N/2} & \mathbf{D}_{N/2} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \cdot \mathbf{W}_N \end{aligned}$$

as in [7] but we do not restrict to the orthogonal case. The Fourier transform $\mathbf{F}_N\mathbf{W}_N^{-1}$ can be then simplified since its result are just the Fourier transforms $\mathbf{F}_N\mathbf{h} = [\mathbf{a}_{N/2}\ \mathbf{c}_{N/2}]^T$ and $\mathbf{F}_N\mathbf{g} = [\mathbf{b}_{N/2}\ \mathbf{d}_{N/2}]^T$ and their time-shifted versions. Writing $\mathbf{A}_{N/2}$, $\mathbf{B}_{N/2}$, $\mathbf{C}_{N/2}$, $\mathbf{D}_{N/2}$ as diagonal matrices with $\mathbf{a}_{N/2}$, $\mathbf{b}_{N/2}$, $\mathbf{c}_{N/2}$, $\mathbf{d}_{N/2}$ on the diagonals, respectively, the factorization can be formulated in a more compact form. For short, we will refer to the matrix built from these diagonal matrices as $\mathbf{P}_N$.

WPT is a generalized version as it iterates the two-channel filter banks in the subband and thus effectively decomposes both high and low frequency bands [8]. Considering the full binary tree results in a completely evenly spaced frequency resolution and

$$\begin{aligned} \mathbf{F}_N &= d^{-1} \cdot \left[\mathbf{P}_N \bigotimes_{q=1}^{D-1} (\mathbf{I}_2 \otimes \mathbf{P}_{N/2^q}) \right] \cdot \left[\mathbf{I}_{2^D} \otimes \mathbf{F}_{N/2^D} \right] \\ &\quad \cdot \left[\bigotimes_{q=1}^{D-1} (\mathbf{I}_2 \otimes \mathbf{W}_{N/2^q}) \mathbf{W}_N \right], \quad (10) \end{aligned}$$

where $\otimes$ represents the Kronecker product. The notation $\otimes_{q=1}^{t}\mathbf{A}_q$ is a shorthand for $\mathbf{A}_1 \otimes \cdots \otimes \mathbf{A}_t$ for $t \geq 1$ and 1 for $t \leq 1$. D is the number of decomposition levels.

3.2. Block-recursive Fourier Transform

The frequency down-sampling as given in Eq. 5 can be elegantly considered by a block-recursive Fourier transform. The proposed scheme considers the FFT of two consecutive blocks of size L and sum the corresponding values to obtain the even channels of a $2L$-point DFT which is equivalent to remove the odd frequencies of the $2L$-point DFT. In general, the DFT of $\mathbf{x} = [x(0), x(1), \dots, x(ML-1)]^T$ can be formulated as

$$\begin{aligned} \mathbf{y} &= \mathbf{F}_{ML}\mathbf{x} \quad (11) \\ \mathbf{y}_q &= \sum_{m=0}^{M-1} \mathbf{F}_L \mathbf{E}_{qm} \mathbf{x}_m, \quad (12) \end{aligned}$$

where $\mathbf{y} = [y(0)\ y(1) \dots y(ML-1)]^T$. $\mathbf{y}_q$ is the q-th channel of the down-sampled column vector $\mathbf{y}$ by M, i.e. it contains the DFT results $y(k)$ at discrete frequencies $k = \{q, q+M, \dots, q+M(L-1)\}$ for $q = \{0, 1, \dots, M-1\}$. $\mathbf{E}_{qm}$ is a diagonal matrix with entries $\{e^{-j\frac{2\pi q(mL)}{ML}}, e^{-j\frac{2\pi q(mL+1)}{ML}}, \dots, e^{-j\frac{2\pi q(mL+(L-1))}{ML}}\}$, and $\mathbf{x}_m = [x(mL), x(mL+1), \dots, x(mL+L-1)]^T$. In our case, the decimation by 2 implies $M = 2$ and $q = 0$.

Finally, the combination of the WPT and block-recursive FFT leads to the following factorization of $(\downarrow 2)\mathbf{F}_N$:

$$(\downarrow 2)\mathbf{F}_N = d^{-1}\left[\mathbf{P}_{N/2}\bigotimes_{q=1}^{D-1}(\mathbf{I}_2 \otimes \mathbf{P}_{N/2^{q+1}})\right] \quad (13)$$
$$\left[\mathbf{I}_{2^D} \otimes (\downarrow 2)\mathbf{F}_{N/2^D}\right]\left[\bigotimes_{q=1}^{D-1}(\mathbf{I}_2 \otimes \mathbf{W}_{N/2^q})\mathbf{W}_N\right].$$

In order to control adaptively the number of FFT blocks to be performed, we can algorithmically characterize the control signals by a vector $\boldsymbol{\gamma}_D(n) = [\gamma_0\, \gamma_1\, \cdots\, \gamma_{2^D-1}]$ with $\gamma_i \in \{0, 1\}$ for the parallel FFT blocks as $\mathbf{I}_{2^D}\boldsymbol{\Gamma}_D(n) \otimes (\downarrow 2)\mathbf{F}_{N/2^D}$, where $\boldsymbol{\Gamma}_D(n)$ is a time-varying diagonal matrix with $\boldsymbol{\gamma}_D(n)$ on its diagonal. Here, $\gamma_i = 0$ implies that the i-th FFT block can be set to zero since its corresponding subband has been characterized to be irrelevant.

The main contribution of applying a WPT-based computation is that WPT leads to a spectral decomposition of the considered signal into subbands. In order to trade-off computational complexity against accuracy, one can now select the subbands of interest ("best subbands") while suppressing the rest. The selection can be either static based on a-priori knowledge about the process or data-driven adapted to time-varying signal. For the last case, we will need a measure to evaluate the relevant subbands among all available and the overhead of these cost functions should be accurate but simple enough in order not to gamble away the savings. It is easy to understand that the primary subbands should give large coefficients in magnitude corresponding to large energy so that the problem reduces to estimate the energy of each subband. Different energy estimation schemes can be considered and are not discussed further in this paper. Our proposed data-driven adaptive scheme to compute the AF-GDTFD of $x(n)$, $n = \{0, 1, \dots\}$ consists now of the following steps:

Algorithm 1 Proposed Adaptive Computation of AF-GDTFD

1: Choose a kernel and compute its eigen-decomposition, the number of relevant eigenvalues $\hat{L}$, precompute all matrices for WPT and compensation network.
2: Start with $\boldsymbol{\Gamma}_D(-L, l \in [1 \dots \hat{L}]) = \mathbf{I}_{2^D}$.
3: Start with n = 0
4: **repeat**
5: **for** $k = 1 \dots \hat{L}$ **do**
6: Perform $\mathbf{V}_k^* \cdot \mathbf{x}(n)$ as given in Eq. 5.
7: Calculate the N-point FFT using the factorization in Eq. 13 and $\boldsymbol{\Gamma}_D(n-1, k)$.
8: Calculate a new $\boldsymbol{\Gamma}_D(n, k)$ based on an application-specific energy estimation procedure.
9: Calculate the spectrogram weighted by λ_k and update the $\mathbf{tfd}_{AF}(n)$.
10: **end for**
11: **until** End of signal

4. COMPLEXITY DISCUSSION

The estimation of operational counts can follow the three separated stages WPT, block-recursive FFT with $M = 2$ and the compensation network. For the comparison, we note that a complex mul can be realized by 3 real muls and 5 real adds. A standard radix-2 N-point FFT requires in total $\frac{N}{2}logN$ complex muls and $2NlogN$ complex adds. Further, we assume for simplicity the orthogonal case, where both filters have the same length M. We introduce also a variable S denoting the subband usage in %.

For the computation of WPT we refer to the lifting scheme which requires $(\frac{N\cdot(M+1)}{2} + 2)D$ real mul and $\frac{N\cdot(M+1)}{2}D$ real adds [9]. For 2^D block-recursive FFTs of length $N/2^D$, we have $S2^D M(\frac{1}{2}\frac{N}{M2^D}log\frac{N}{M2^D})$ complex muls. The number of complex adds consists of $S2^D M(2\frac{N}{M2^D}log\frac{N}{M2^D})$ for the FFT, $\frac{N}{2}$ for the averaging, and an additional part due to the complex multiplication.

For the cost of the compensation network we have to differ between the case $D = 1$ and $D \geq 2$. For the first case, the matrix $\mathbf{P}_{N/2}$ has two non-zero terms in each row resulting in $S2\frac{N}{2}$ complex muls. As above, in addition to the number of adds due to complex multiplications, there is one complex addition per row yielding another $S\frac{N}{2}$ complex adds. If $D \geq 2$, note that the factor S only influences the first and last compensation stage. The operational counts for muls are twice of that for $D = 1$. The first stage requires $2^{D-1} \cdot \frac{N}{2^D} \cdot \lfloor S2^{D-1} \rfloor$ complex adds where 2^{D-1} is the number of $\mathbf{P}_{N/2^{q+1}}, q = D-1$ with size $\frac{N}{2^D} \times \frac{N}{2^D}$. For the last stage, one has again $S\frac{N}{2}$ complex adds. For the $D-2$ mid-stages, there are $(D-2)\frac{2N}{M}$ complex muls and $(D-2)\frac{N}{2}$ complex adds.

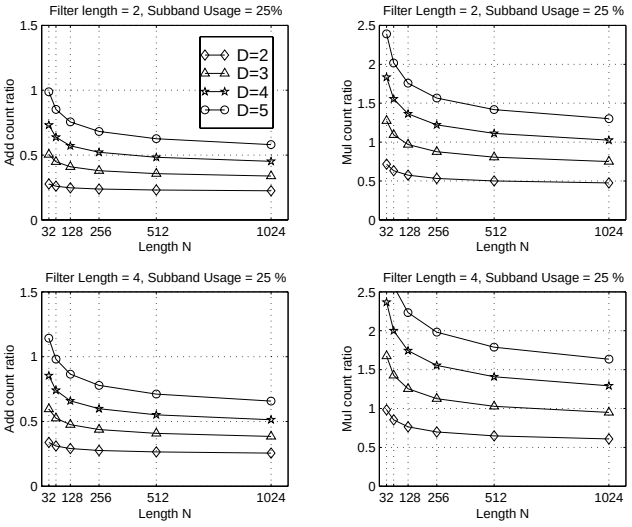

Figure 1: Normalized cost of Eq. 13 versus standard radix-2 FFT followed by $(\downarrow 2)$.

In Fig.1, the normalized computational cost for the filter length 2 (Haar) and 4 is given if a subband usage of 25% is assumed. With a moderate number of decomposition levels (up to 3), the proposed approach requires especially for long signal block clearly less computation operation as using the standard approach for the given approximation degree. Further, the number of real additions is reduced much stronger than the multiplication due to the com-

pensation network which puts the highest burden on the total cost. This is also the reason why more than 3 decomposition levels are not recommended although a sharper subband selection would be possible. From this point of view, it is also clear that the proposed scheme is advantageous in terms of computational counts if only approximation is involved. The selection of filter length and decomposition levels depends on a-priori knowledge about the process or expected value of subband usage.

5. EXAMPLE

The proposed approach has been applied to a synthetic signal consisting of one Dirac pulse, two simultaneous sinusoidal pulses, a chirp, a Gaussian pulse and again two Dirac pulses. A discrete cone-shaped kernel $\psi(n,m) = e^{-2\alpha m^2}$ for $|n| \leq \frac{|m|}{2}$, $|m| \leq L-1$, with $L = 128$ was used (i.e. $N = 256$). α was so chosen that $\psi(n, \pm L) = 0.01$. $\hat{L}$ was set to 6 with $\lambda_{1\ldots6} =$ 31.1277, -28.7581, -4.0667, 3.4149, -0.7395, 0.3639. For the WPT we chose Daubechies 4-tap filter and two decomposition levels. The energy estimation is based on the sum of absolute WPT coefficients. γ_i, $i \in \{0,1,2,3\}$ is set to one if the estimated energy of its corresponding subband exceeds the adaptive threshold. The strategy to set the threshold in this example is based on the knowledge of the chosen eigenvalues. In each time step at the processing with λ_l, the maximum estimated subband energy is scaled by $1-|\lambda_l|/\sum_{k=1}^{\hat{L}}|\lambda_k|$ and serves as a hard threshold for the subband selection in the next time step. The result is shown in Fig. 2. It is apparent that although only about 20.5% of subbands on the average are selected for usage, the visualization is still sharp enough to follow the time-varying signal in the AF-GDTFD. Based on Fig. 1, the complexity has been reduced to at least 28% and 70% for the number of real adds and muls, respectively. The Frobenius norm of the average error for each time-frequency sample between the AF-GDTFDs in Fig. 2 takes on $5.2 \cdot 10^{-6}$.

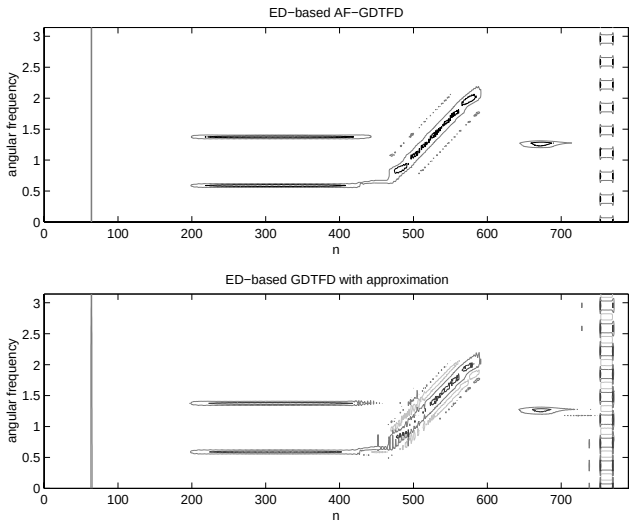

Figure 2: On the top is the AF-GDTFD computed using the first six largest eigenvalues and standard FFT. Below is the result if the additional approximation is performed using the proposed approach.

6. CONCLUSIONS

In this paper, we have proposed a data-driven scheme to accelerate the computation of discrete AF-GDTFD targeting the time-frequency representation of long signals and/or real-time systems. The main idea is to follow extensively the concept of approximation and incremental refinement in order to trade-off computational requirements against accuracy. Our approach involves subband filtering on the basis of WPT to decompose the input signal into different spectral subbands (not restrictive to orthogonal wavelets). FFT blocks of smaller size are then applied to the outputs of WPT and the results are recombined together through a compensation network. A block-recursive scheme inside the FFT blocks again reduces the computational complexity of the following compensation network which constitutes the largest computational cost for N less than 256. Different energy estimation scheme can be utilized to select only relevant subbands in an adaptive or static manner. A detailed analysis of complexity shows that even savings in the order of more than 50% is realizable if a low subband usage can be achieved (as is the case for applications in monitoring of mechanical processes) and a moderate number of WPT decomposition levels is applied. Furthermore, the proposed scheme is quite attractive in terms of VLSI due to modular structure, local connections and re-usability (parallel FFT blocks of smaller size) as well as stream processing (WPT and block-recursive computation of FFT). These aspects are currently under investigation.

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