ROBUST ENVELOPE-CONSTRAINED FILTER DESIGN WITH LAGUERRE BASES

C. H. Tseng, Z. Zang, K. L. Teo, and A. Cantoni

Australian Telecommunications Research Institute,
Curtin University of Technology, GPO Box U 1987 Perth, WA, 6845 Australia.

ABSTRACT

The envelope-constrained filtering problem is concerned with the design of a filter such that the noise enhancement is minimized while the noiseless filter response stays within an envelope. Naturally, the optimum filter response to the prescribed input signal tends to touch the output boundaries at some points. Consequently, any disturbance to the prescribed input signal could result in the output constraints being violated. In this paper, we formulate a semi-infinite constrained optimization problem in which the margin of the constraint robustness of the filter is maximized. Using a smoothing technique, it is shown that the solution of the optimization problem can be obtained by solving a sequence of strictly convex optimization problems with integral cost.

1. INTRODUCTION

Consider a linear time-invariant (LTI) filter with impulse response to process a given input signal which is corrupted by additive random noise as shown in Figure 1(a). The objective is to design a filter such that its squared L^2 norm is minimized while its noiseless output corresponding to a specified input signal stays within the pre-specified lower and upper boundaries, see Figure 1(b). The choice of the objective function is due to the fact that the output noise power is proportional to the squared L^2 norm of the filter to be designed [3]. Thus the continuous-time envelope-constrained filtering problem may be posed as follows:

$$\begin{array}{ll} \min & \|u\|^2 \\ \text{subject to} & \epsilon^-(t) \leq \psi(t) \leq \epsilon^+(t), \forall t \in [0, \infty). \end{array}$$

Traditionally, problems of this type were often handled by least mean square (LMS) approach. However, it is known that the EC filtering approach is more relevant than the "soft" LMS approach in a variety of signal processing fields [1, 3, 4].

The EC filtering problem was first posed in early 1970s [3, 4]. Since then, various methods for solving this problem have been reported in the literatures [3, 8]. Naturally, the

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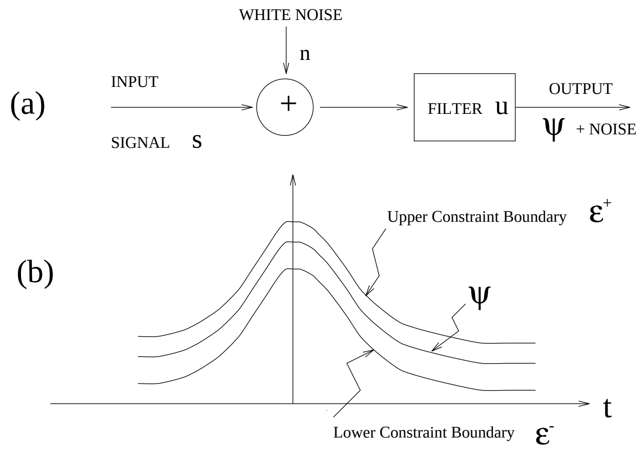

Figure 1: (a)Block diagram. (b)Pulse shaping envelope.

response of the optimum filter to the prescribed input signal tends to touch the output boundaries at some points. Consequently, any disturbance to the prescribed input signal or errors in the implementation of the optimal filter could result in the output constraints being violated. Thus, it is of practical importance to address the robustness issue of the filter in that the minimum distance between the output response and the output envelope is maximized.

2. PROBLEM FORMULATION

2.1. EC Filter with Laguerre Bases

Consider the block diagram of the EC filter involving Laguerre network in Figure 2. The time-domain Laguerre function with an adjustable pole $p > 0$ is given by

$$L_j^p(t) = \sum_{i=0}^{j} \begin{pmatrix} j \\ j-i \end{pmatrix} \frac{(-2pt)^i}{i\,!} \sqrt{2p} e^{-pt}, \quad j = 0, 1, 2, \ldots$$

It is known that the Laguerre sequence $\{L_j^p\}_{j=0}^{\infty}$ forms a uniformly bounded orthonormal basis for the Hilbert space $L^2([0,\infty))$(cf. [6]). Thus, any $u(t) \in L^2([0,\infty))$ can be represented as: $u(t) = \sum_{j=0}^{\infty} x_j L_j^p(t)$ where $x_j = \langle u, L_j^p \rangle$,

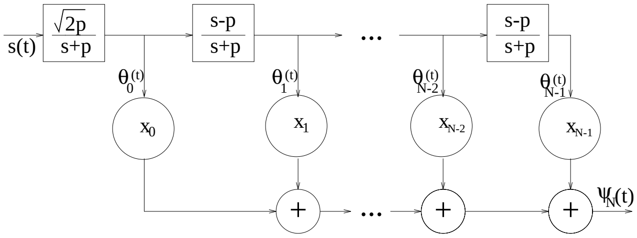

Figure 2: An EC continuous-time Laguerre network

$j = 0, 1, \ldots$ are the Laguerre-Fourier coefficients. We consider only those filters $u_N(t)$ whose impulse response are approximated by: $u_N(t) = \sum_{j=0}^{N-1} x_j L_j^p(t)$. The corresponding filter output is: $\psi_N(t) = \int_0^T u_N(\tau)s(t-\tau)d\tau = \Theta^T(t)\mathbf{x}$ where $\Theta^T(t) = [\theta_0(t), \ldots, \theta_{N-1}(t)]$. Then $\|u_N\|^2 = \mathbf{x}^T\mathbf{x}$ where $\mathbf{x}^T = [x_0, \ldots, x_{N-1}]$. Thus from Figure 2, the continuous EC filtering problem can be expressed as the QP problem below:

Problem P_0:

$$\begin{array}{ll} \min & \|\mathbf{x}\|^2 = \mathbf{x}^T\mathbf{x}, \ \ \mathbf{x} \in \Re^N \\ \text{subject to} & \epsilon^-(t) \le \psi_N(\mathbf{x}, t) \le \epsilon^+(t), \forall t \in [0, T]. \end{array}$$

It is clear that the objective function is strictly convex and the constraint set is convex. Thus, Problem P_0 admits a unique optimal solution provided the constraint set is nonempty. To avoid the trivial solution $u_N(t) = 0$ (i.e, $\mathbf{x} = 0_N$), we impose the following assumption.

Assumption 2.1 *There exists at least one point in the output mask at which the upper and lower mask boundaries have the same sign.*

2.2. Constraint Robustness Formulation

In this subsection, we present a technique for providing a guard band on the output mask. For a given filter coefficient $\mathbf{x}$, let us define

$$\left\{ \begin{array}{lcccc} [\phi^+(\mathbf{x})](t) & = & \psi_N(\mathbf{x}, t) - \epsilon^+(t) & \le & 0 \\ [\phi^-(\mathbf{x})](t) & = & \epsilon^-(t) - \psi_N(\mathbf{x}, t) & \le & 0. \end{array} \right.$$

To quantify the notion of robustness, we define the constraint robustness margin as:

$$\sigma(\mathbf{x}) \quad = \quad \min\left\{ \min_t [-\phi^+(\mathbf{x})](t), \min_t [-\phi^-(\mathbf{x})](t) \right\}.$$

The feasible region of Problem P_0 can now be expressed in terms of the robustness margin as: $\mathcal{F} = \{\mathbf{x} \in \Re^N : \sigma(\mathbf{x}) \ge 0\}$. Note that if $\sigma(\mathbf{x}) > 0$, the minimum distance of the output $\psi_N(\mathbf{x}, t)$ from the output mask is at least equal to $\sigma(\mathbf{x})$. Therefore, we say that the filter u_N is robust with constraint robustness margin $\sigma(\mathbf{x})$. In practice, it may be necessary to have a larger constraint robustness margin over certain intervals. In this case, a weighting function β can be used to achieve the purpose. More specifically, we define the weighted constraint robustness margin as follows:

$$\sigma_\beta(\mathbf{x}) = \min\left\{ \min_t \frac{[-\phi^+(\mathbf{x})](t)}{\beta(t)}, \min_t \frac{[-\phi^-(\mathbf{x})](t)}{\beta(t)} \right\}$$

where β is a positive continuous weighting function which is normalized so that it attains a minimum of unity. Then the EC filtering problem with robustness constraint may be formulated as the following constrained optimization problem.

Problem Q:

$$\begin{array}{ll} \max & \sigma_\beta \\ \text{subject to} & \epsilon^-(t) + \beta(t)\sigma_\beta \le \psi_N(\mathbf{x}, t) \le \epsilon^+(t) - \beta(t)\sigma_\beta \\ & \|\mathbf{x}\|^2 \le (1+\delta)\|\mathbf{x}^*\|^2, \sigma_\beta \ge 0, \forall t \in [0, T] \end{array}$$

where $\delta > 0$ is a constant which specifies the allowable amount of increase of the output noise power and $\mathbf{x}^*$ denote the optimal solution of Problem P_0. Define the feasible region of EC filtering problem with robustness constraint as follows:

$$\begin{aligned} \mathcal{F}_{\sigma_\beta} \quad = \quad & \{x \in \Re^N : |\psi_N(\mathbf{x}, t) - d(t)| \le \epsilon(t) - \beta(t)\sigma_\beta, \\ & \|\mathbf{x}\|^2 \le (1+\delta)\|\mathbf{x}^*\|^2, \forall t \in [0, T]\} \end{aligned} \tag{2.1}$$

where $d(t) \triangleq \frac{\epsilon^+(t)+\epsilon^-(t)}{2}$ is a desired pulse shape, and $\epsilon(t) \triangleq \frac{\epsilon^+(t)-\epsilon^-(t)}{2}$ is an error tolerance band about $d(t)$. The following proposition characterizes the sensitivity of the feasible point in the set $\mathcal{F}_{\sigma_\beta}$ for different values of σ_β.

Proposition 2.2
For any given σ_β^1 and σ_β^2 such that $\mathbf{x}_{\sigma_\beta^1} \in \mathcal{F}_{\sigma_\beta^1}$ and $\mathbf{x}_{\sigma_\beta^2} \in \mathcal{F}_{\sigma_\beta^2}$. If $0 < \sigma_\beta^1 \le \sigma_\beta^2$ then $\mathcal{F}_{\sigma_\beta^2} \subseteq \mathcal{F}_{\sigma_\beta^1}$ and $\|\mathbf{x}_{\sigma_\beta^1}\| \le \|\mathbf{x}_{\sigma_\beta^2}\|$.

The above proposition indicates that the optimal σ_β must be a bounded positive constant. Otherwise, it is no solution for problem Q at any finite σ_β. In addition, a feasible point can be found to achieve a bigger constraint robustness margin (i.e, a tighter output mask), but at the expense of the increased noise gain $\|\mathbf{x}\|^2$. Thus, there is always a compromise between the tightness of the output mask and the output noise gain of the filter.

3. ROBUST ENVELOPE-CONSTRAINED FILTER

Consider Problem Q. Let the constraint be defined as:

$$\left\{ \begin{array}{lcl} g_1(\mathbf{x}, t|\sigma_\beta) & \triangleq & \beta(t)\sigma_\beta - \psi_N(\mathbf{x}, t) + \epsilon^-(t) \\ g_2(\mathbf{x}, t|\sigma_\beta) & \triangleq & \beta(t)\sigma_\beta + \psi_N(\mathbf{x}, t) - \epsilon^+(t). \end{array} \right.$$

Clearly, for a given $\sigma_\beta \geq 0$, $g_1(\mathbf{x}, t|\sigma_\beta)$ and $g_2(\mathbf{x}, t|\sigma_\beta)$ satisfy the following conditions:
i) $g_j(\mathbf{x}, t|\sigma_\beta)$, $j = 1, 2$, are continuous in $t \in [0, T]$, and for each $\mathbf{x}$, $\frac{\partial g_j(\mathbf{x},t|\sigma_\beta)}{\partial t}$ is piecewise continuous in $t \in [0, T]$;
ii) $g_j(\mathbf{x}, t|\sigma_\beta)$, $j = 1, 2$ are continuously differentiable with respect to $\mathbf{x}$ for almost all $t \in [0, T]$.

Finding the maximum σ_β that solves Problem Q is equivalent to finding the maximum σ_β for which the set $\mathcal{F}_{\sigma_\beta}$ remains non-empty. Thus, we can start with σ_β and check if the corresponding set $\mathcal{F}_{\sigma_\beta}$ remains nonempty. If it does, we can increase the value of σ_β and repeat the process. On the other hand, if the set $\mathcal{F}_{\sigma_\beta}$ becomes empty for a given σ_β, it follows from Proposition 2.2 that the value of σ_β should be reduced. To check the feasibility of the set $\mathcal{F}_{\sigma_\beta}$, we use the idea reported in [5] for functional inequality constraints. We construct the following constraint convex optimization problem.

Problem SQ:

$$\min \left\{ J_\xi(\mathbf{x}) = \int_0^T \left(\Phi_\xi(g_1(\mathbf{x}, t|\sigma_\beta)) + \Phi_\xi(g_2(\mathbf{x}, t|\sigma_\beta)) \right) dt \right\}$$
$$\text{subject to } ||\mathbf{x}||^2 \leq (1+\delta)||\mathbf{x}_0||^2$$

where $\Phi_\xi(\cdot)$ is defined by

$$\Phi_\xi(g_j) = \begin{cases} 0 & \text{if } g_j < -\xi \\ \frac{(g_j+\xi)^2}{4\xi} & \text{if } -\xi \leq g_j \leq \xi \\ g_j & \text{if } g_j > \xi \end{cases}$$

Note that $\Phi_\xi(g_j)$ possess the following properties:
i) $\Phi_\xi(g_j)$ is once continuously differentiable and piecewise twice continuously differentiable.
ii) $\Phi_\xi(g_j)$ is convex and monotonically non-decreasing.

From Propositions 3 and 4 of [5] which establish the necessary and sufficient conditions for feasibility of the set $\mathcal{F}_{\sigma_\beta}$, it is clear that finding a feasible point $\mathbf{x}$ for a given σ_β is equivalent to solving Problem SQ. Moreover, Problem SQ is solvable by any gradient-based quasi-Newton method. The following algorithm can be used to determine whether a particular σ_β is feasible.

Algorithm 3.1 *Choose a $\sigma_\beta \geq 0$ and an $\xi > 0$. At each iteration of the quasi-Newton algorithm, we insert the following steps for solving Problem SQ:*
1. If $J_\xi(\mathbf{x}^k) > \frac{\xi T}{2}$ then go to step 3; otherwise, go to step 2.
2. Check if the constraints of Problem Q are satisfied. If so, go to step 4; otherwise, go to step 3.
3. Continue the next iteration of quasi-Newton method to obtain $\mathbf{x}^{k+1}$ and $J_\xi(\mathbf{x}^{k+1})$. Set $k = k+1$, and go to step 1.
4. Stop. $\mathbf{x}^k$ is a feasible point in $\mathcal{F}_{\sigma_\beta}$.

By means of solving Problem SQ, Problem Q can be easily solved by using the combination of the golden section search method [7] and any quasi-Newton method.

Algorithm 3.2 *Set $\sigma_{\beta 0} = 0$, choose a $\sigma_{\beta t} > 0$, and assign the iteration accuracies $\gamma_1 > 0$ and $\gamma_2 > 0$ for σ_β and $(1+\delta)||\mathbf{x}^*||^2 - ||\mathbf{x}^k||^2$, respectively.*
1. Determine if $\mathcal{F}_{\sigma_{\beta t}} = \emptyset$ by Algorithm 3.1.
2. If so, go to step 3; otherwise, set $\sigma_{\beta 0} = \sigma_{\beta t}$ and $\sigma_{\beta t} = 2\sigma_{\beta t}$, and then go to step 1.
3. Set
$\bar{\sigma}_{\beta 0} = \sigma_{\beta 0} + (1-\lambda)(\sigma_{\beta t} - \sigma_{\beta 0})$; $\bar{\sigma}_{\beta t} = \sigma_{\beta 0} + \lambda(\sigma_{\beta t} - \sigma_{\beta 0})$ where $\lambda \simeq 0.618$ is the golden section ratio.
4. If $\mathcal{F}_{\bar{\sigma}_{\beta 0}} = \emptyset$, set $\sigma_{\beta t} = \bar{\sigma}_{\beta 0}$ and go to step 6; otherwise, set $\sigma_{\beta 0} = \bar{\sigma}_{\beta 0}$ and go to step 5.
5. If $\mathcal{F}_{\bar{\sigma}_{\beta t}} = \emptyset$, set $\sigma_{\beta t} = \bar{\sigma}_{\beta t}$ and go to step 6; otherwise, set $\sigma_{\beta 0} = \bar{\sigma}_{\beta t}$ and go to step 6.
6. If $\sigma_{\beta t} - \sigma_{\beta 0} \leq \gamma_1$ and $(1+\delta)||\mathbf{x}^*||^2 - ||\mathbf{x}^k||^2 \leq \gamma_2$, set $\sigma_{\beta_{max}} = \sigma_{\beta 0}$ and stop; otherwise, go to step 3 and replace k by $k+1$.

Theorem 3.3 *[9] Algorithm 3.2 is guaranteed to locate the optimal solution $\sigma_{\beta_{max}}$ of Problem Q to within a given interval of uncertainty in a finite number of iterations.*

4. NUMERICAL RESULTS

In this section, we consider the equalization of a digital transmission channel involving a coaxial cable operating at the DSX3 rate (44.736 Mb/s) [2]. The design objective is to find an equalizer which takes the impulse response of a coaxial cable with a loss of 30 dB and produces an output which lies within the DSX3 pulse template, see Figure 3. To have a good representation of the input signal, the analog input signal is sampled every $\frac{\beta}{32}$ time unit over $[0, T]$ where $T = 32\beta$. The constraint robustness problem is solved with a weighting function $\beta(t) = \frac{\epsilon^+(t) - \epsilon^-(t)}{2}$ where $\epsilon^+(t)$ and $\epsilon^-(t)$ are defined in Problem P_0. In our simulation, the following parameters are applied to Algorithm 3.1 and Algorithm 3.2: $N = 8$, $p = 14$, $||\mathbf{x}^*||^2 = 59.39677$, $\xi = 10^{-3}$, $\sigma_{\beta 0} = 0$, $\sigma_{\beta t} = 0.1$ and $\delta = 1.0$, which implies that we are prepared to accept an additional 100% increase in output noise power for an improved robustness. The simulation results are depicted in Figures 4-5. It is clear from Figure 4 that Algorithm 3.2 is guaranteed to locate the maximum constraint robustness margin, $\sigma_{\beta max} = 0.3957$, in a finite number of iteration in terms of adjusting σ_β and satisfing constraints of Problem Q. Figure 5 illustrates that if it allows more output noise power, then the filter output response will be forced to be closer to the desired pulse shape.

5. CONCLUSION

In this paper, the robustness of continuous-time envelope-constrained filtering problem is studied. A simple yet efficient algorithm has been proposed to solve the problem so that the constraint robustness margin is maximized. The key success to the proposed algorithm is to seek a feasible point to $\mathcal{F}_{\sigma_\beta}$ where $0 \leq \sigma_\beta \leq \sigma_{\beta_{max}}$. A smoothing technique is then applied to solve this problem which converts the semi-infinite constrained problem into a strictly convex constrained problem with integral cost.

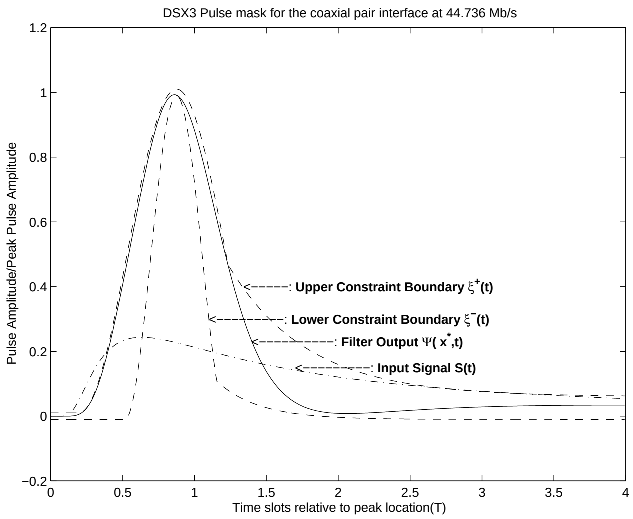

Figure 3: The filter output response fits into envelope with non-constraint robustness.

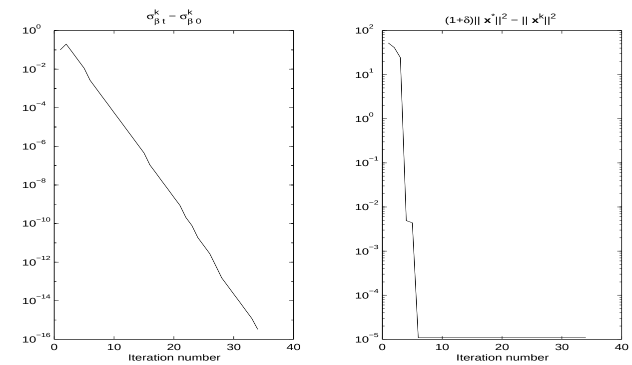

Figure 4: The convergence results of the robust envelope-constrained filtering problem.

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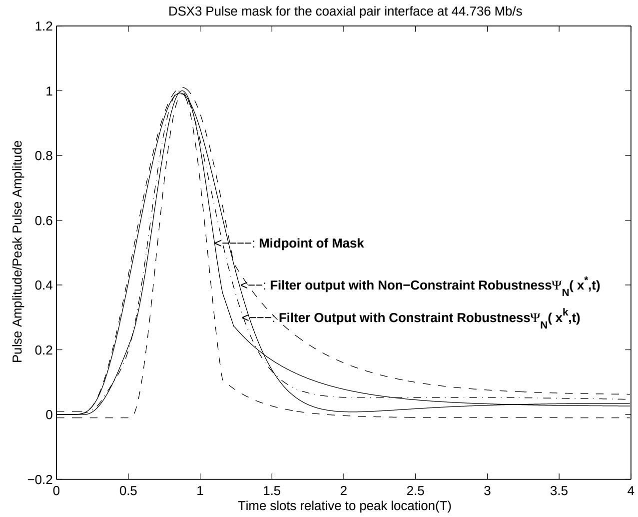

Figure 5: The filter output response fits into envelope with constraint robustness at $\delta = 1.0$.

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