TWO-DIMENSIONAL PHASE RETRIEVAL USING A WINDOW FUNCTION

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ABSTRACT

This paper considers two-dimensional phase retrieval using a window function. First, we address the uniqueness and reconstruction of a two-dimensional signal from the Fourier intensities of the three signals: the original signal, the signal windowed by a window $w(m,n)$, and the signal winowed by its complementary window $w_c(m,n) = 1 - w(m,n)$. Then we consider the phase retrieval without a complementary window. We develop conditions under which a signal can be uniquely specified from the Fourier intensities of the original signal and the windowed signal by $w(m,n)$. We also present a reconstruction algorithm.

1. INTRODUCTION

Phase retrieval is concerned with the reconstruction of a signal or Fourier transform phase from its Fourier transform magnitude. Since the problem does not have a unique solution in general, researchers have tried many ways by providing information of the signal *a priori* or constraining the properties of the signal.

One approach is using window functions. In this approach researcher have tried several methods. For example, Nawab *et al.* considered the reconstruction of a signal from its Fourier transform intensities of its short time sections [1]. Gonsalves derived differential equations from the Fourier intensities of the original signal and the windowed signal by a weighted aperture and considered the reconstruction of the signal [2]. Wood *et al* discovered the fact that there is a certain relationship between the zeros of a signal and the zeros of the signal weighted by an exponential window. They presented an example showing that the zeros can be retrieved from the zeros of the autocorrelations of the original signal and the windowed signal [3]. Nakajima considered the numerical reconstruction of the original signal of this problem and developed two algorithms [4][5]. The author, also, presented several conditions under which a one dimensional signal $x(n)$ can be uniquely specified from the Fourier intensity of three signals; the signal itself, a signal windowed by a window function $w(n)$, and the signal windowed by the complementary window function $w_c(n) = 1 - w(n)$ [6].

In this paper, we consider two-dimensional phase retrieval using a window function. This paper is organized as follows. In the uniqueness section, we present uniqueness conditions under which a two-dimensional signal can be uniquely specified from the Fourier intensities of the signal and the windowed signal with or without the Fourier intensity of the signal windowed by the complementary window. In the reconstruction section, we present two reconstruction algorithms and compare the performance of the algorithms.

2. UNIQUENESS

In [6], the one-dimensional phase retrieval from three Fourier intensities is considered: the Fourier intensity of the original signal, the Fourier intensity of the signal windowed by a window $w(n)$ and the signal windowed by its complementary window $w_c(n) = 1 - w(n)$. And its uniqueness is summarized in Theorem 3 of [6]. We begin with the extension of Theorem 3 to two-dimensional case.

Theorem 1. Let $x(m,n)$ be a real, non-negative, finite-support sequence with a minimum nonzero region of support $R(M,N)$ and let $w(m,n)$ and $w_c(m,n)$ be complementary windows to each other defined in $R(M,N)$. Let $y(m,n)$ and $z(m,n)$ be windowed signals of $x(m,n)$ such that

$$y(m,n) = x(m,n)w(m,n) \text{ and } z(m,n) = x(m,n)w_c(m,n).$$

Suppose the z-transform of the autocorrelation of $y(m,n)$ and $z(m,n)$ do not have common divisor. Then the signal $x(m,n)$ can be uniquely defined from the Fourier intensities $|X(\varpi_1,\varpi_2)|^2$, $|Y(\varpi_1,\varpi_2)|^2$, and $|Z(\varpi_1,\varpi_2)|^2$ if

$$w(m,n) \neq w(M-1-m, N-1-n).$$

The proof of this theorem is given in [7]. According to Theorem 1, we need three Fourier intensities to find a signal satisfying the conditions of Theorem 1. For two-dimensional signals, however, we have found that two Fourier intensities are enough for solving the phase retrieval problem in some cases.

Two or higher dimensional signals have a different property compared with one-dimensional signals. According to Gauss' Fundamental Theorem of Algebra, any one-dimensional polynomial can be factorized into the product of first order polynomials, which means that one dimensional signals can not be specified uniquely from its Fourier intensities to within trivial ambiguities. For two or higher dimensional signals, however, it is proved that the set of two or higher dimensional reducible polynomials has zero measure [8]. This means almost all the z-transforms of two or higher dimensional signals are irreducible, and therefore, two or higher dimensional signals are uniquely specified from its Fourier intensities only to within translation, signal, and space reversal ambiguities [9]. Exploiting this property and adding some constraints, we may uniquely specify a two-dimensional signal from two Fourier intensities.

Theorem 2. Let $x(m,n)$ be a non-negative real two-dimensional signal having Region of Support $R(M,N)$ and have an irreducible z-transform. Let $w(m,n)$ be a window function whose R.O.S. is $R(M,N)$ and any translation of $w(m,n)$ is not included in $R(M,N)$. Let $y(m,n)$ be defined as $y(m,n) = x(m,n)w(m,n)$ and has an irreducible z-transform. Suppose

$$w(m,n) \neq w(M-1-m, N-1-n),$$

then the original signal $x(m,n)$ can be determined uniquely from the Fourier intensity of $|X(\varpi_1,\varpi_2)|^2$ and $|Y(\varpi_1,\varpi_2)|^2$.

A short justification is as follows. Since $y(m,n)$ has an ireducible z-transform, this signal can be determined uniquely from its Fourier transform intensity to within three trivial ambiguities such as translation, sign reversal, or space reversal. However, since $y(m,n) = x(m,n)w(m,n)$ and $w(m,n) \neq w(M-1-m, N-1-n)$, $y(m,n)$ is non-negative and has different region of support other than $y(M-1-m, N-1-n)$, which removes the ambiguities of sign reversal and space-reversal. Furthermore, the assumption that the translation of $w(m,n)$ is not included in $R(M,N)$ justifies that there is no ambuguity of translation. Proof of Theorem 2 can be found in [7]. Although the assumption that $x(m,n)$ and $y(m,n)$ have irreducible z-transform is not true for all the two-dimensional signals, it is proved in [8] that this asumption is very likely and reasonable.

3. RECONSTRUCTION

The reconstruction of a signal satisfying the uniqueness condition of Theorem 1 may be done by the extension of the algorithm in [6]. Figure 1 is a two-dimensional version of the reconstruction algorithm and Figure 2 is an example that shows the performance of the algorithm when applied to a two-dimensional image signal. Figure 2 (a) is a 128x128 point original image and (b) is the window function given as

$$w(m,n) = \begin{cases} o & if \quad (m,n) \in [10,110] \times [10,110] \\ 1 & \qquad Otherwise \end{cases}$$

© is the result after 100 iterations. As we can see in the figure, the reconstructed signal contains much of its properties and thus the signal can be reconstructed uniquely as iteration goes on.

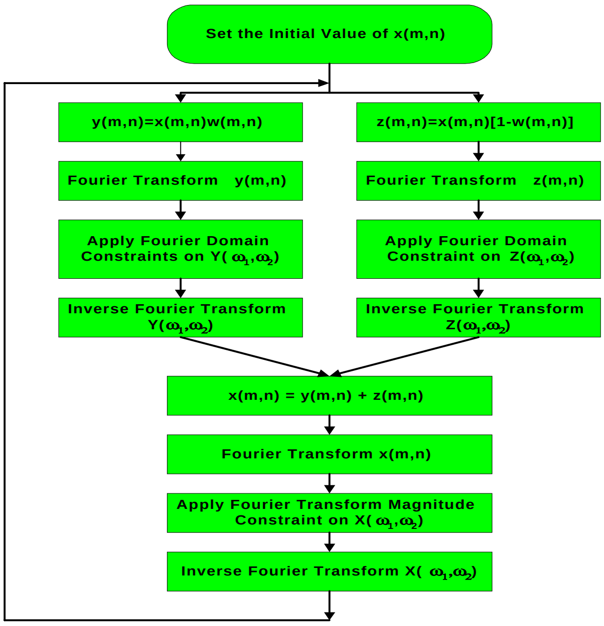
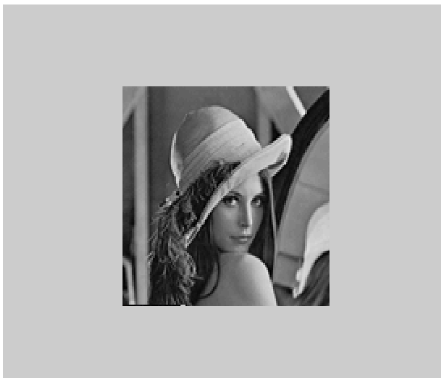
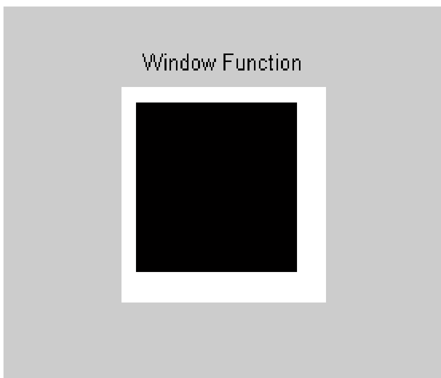


Figure 1. Reconstruction Algorithm #1 for Theorem 1



(a) Original Image



(b) Window function

(c) Reconstructed image after 100 iterations

Figure 2 Reconstruction example using the algorithm in Figure 1

Figure 3 is the block diagram of the proposed algorithm for the reconstruction of a signal satisfying the conditions of Theorem 2. This algorithm is composed of the concatenation of two independent Gerchberg-Saxton algorithms. The result image of the first algorithm is used as an initial value for the second algorithm. If we run more iterations in the first algorithm, then we can get a better initial conditions for the second algorithm, which affects the performance of the second algorithm and the overall algorithm in turn. Figure 4 is an example which shows the performance of Algorithm #2. The conditions for this example is the same as those used in Figure 2. As we can see in this example, after the same number of iterations, this reconstructed image has more features of the original signal than the signal in Figure 2, so we can say that Algorithm #2 has better performance than Algorithm #1. The reason seems to come from the fact that in Algorithm #2 we can avoid the stagnation problem.

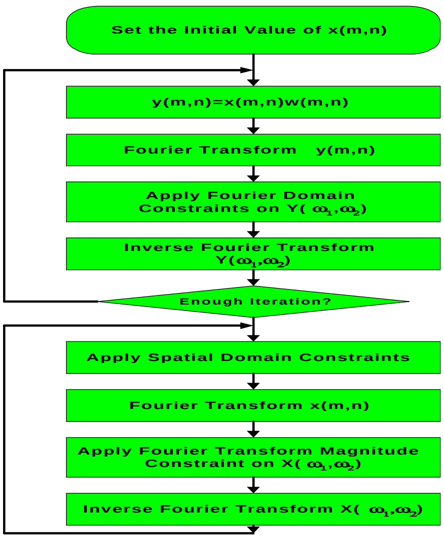


Figure 3 Reconstruction Algorithm #2 for Theorem 2

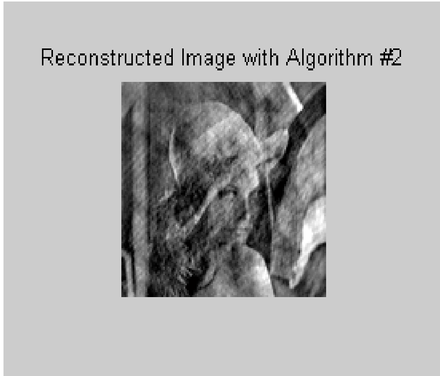


Figure 4 Reconstructed image using Algorithm #2 after 100 iterations

In Figure 5 the mean squared error of the two algorithms, which shows the relative performance between the two algorithms, is given. The mean squared error is defined as

$$MSE = \frac{1}{MN} \sum_{(m,n)\in R(M,N)} [x_r(m,n) - x_o(m,n)]^2 ,$$

where $x_r(m,n)$ is the reconstructed value, $x_o(m,n)$ is the original value.

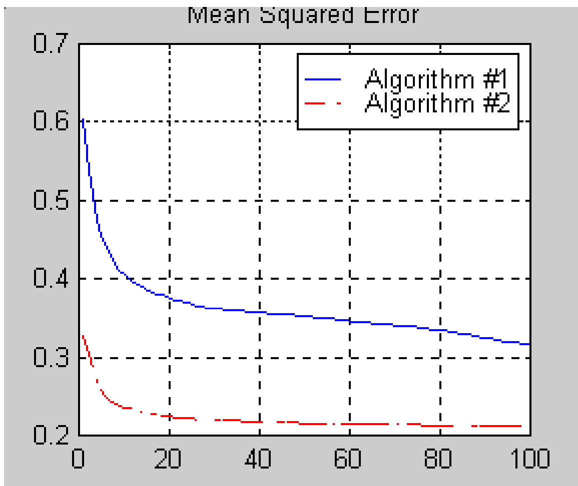


Figure 5 Comparison of the Mean-Squared-Error of the two algorithms

In Figure 6, we present a reconstruction result using a symmetric window after the same 100 iterations. Although this case does not satisfy the uniqueness conditions in Theorem 2, we found that, in many examples including this, Algorithm #2 could find the desired signal where as Algorithm #1 usually failed to find the desired signal.

4. SUMMARY

In this paper, we addressed the signal reconstruction from its Fourier transform intensities using a window function. We considered two cases of two-dimensional phase retrieval. The one is the case where we have three Fourier intensities; the

Fourier intensities of the desired signal, the signal windowed by a window, and the signal windowed by its complementary window. The other is the case when we have the Fourier intensity of the desired signal and the Fourier intensity of the windowed signal. We developed conditions under which a two-dimensional signal can be specified uniquely from the given conditions and presented two reconstruction algorithms. We found that Algorithm #2 for two Fourier intensity case has better performance than Algorithm #1 fir the three Fourier intensity case.

Since in Astronomy or Optics, the phase retrieval problem arises and windows are very popularly used, we hope that the results here can be used in Optics or Astronomy, especially for the performance enhancement of a ground-based telescope..

5. REFERENCES

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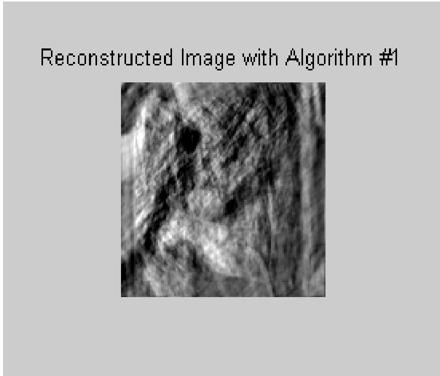

(a) Reconstructed Image Using Algorithm #1

(b) Reconstructed Image Using Algorithm #2

Figure 6 Reconstruction Using a Symmetric Window