A NEW UNITARY ESPRIT-BASED TECHNIQUE FOR DIRECTION FINDING

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ABSTRACT

A new pseudo-noise resampling technique is proposed to mitigate the effect of outliers in Unitary ESPRIT. This scheme improves the performance of Unitary ESPRIT in unreliable situations, where the so-called reliability test has a failure. For this purpose, we exploit a pseudo-noise resampling of a failed Unitary ESPRIT estimator with a censored selection of "successful" resamplings recovering the non-failed outputs of the reliability test.

1. INTRODUCTION

Unitary ESPRIT is a low-complexity modification of conventional ESPRIT formulated in terms of real-valued computations [1], [2]. The final step of Unitary ESPRIT involves a special test [1] showing whether the obtained Direction Of Arrival (DOA) estimate is reliable. In case this test (in what follows referred to as the *reliability test*) has a failure, the final step of the algorithm yields a complex conjugate pair of eigenvalues instead of a real one as in the non-failed case. The failed situation can be interpreted as an *outlier* corresponding to unresolved signal arrivals [1]. In the case of a failed reliability test, it is recommended in [1] and [2] to restart the algorithm with more reliable measurements or to use more snapshots when estimating the covariance matrix. However, in many practical situations, neither more reliable measurements nor additional data snapshots are available.

In this paper, we propose another approach to mitigate such type of outliers. Our approach does not require any additional data, i.e., it exploits exactly the same data snapshots as the failured estimate itself. Instead of unavailable additional data, it utilizes synthetically generated data as in the bootstrap technique. The key idea of our approach is to use a pseudo-noise resampling for eliminating the failure and for recovering the outlier-free performance. For this purpose, we exploit a censored selection of "successful" resamplings for which the reliability test has been passed in the sense that the final step of Unitary ESPRIT yields a real eigenvalue pair. Our technique uses the main idea of the estimator bank approach [3], though we develop and apply another, more general type of resampling [4]. Another important difference is that the standard estimator bank approach [3] requires *a priori* knowledge of source localization sectors, whereas the reported technique is free of such a limitation.

Supported in part by DFG under Grant Bo 568/22-1, INTAS under Grant INTAS-93-642-Ext, and the Russian Foundation of Basic Research under Grant 96-02-19462.

2. SIGNAL MODEL

Let a Uniform Linear Array (ULA) be composed of M sensors and let it receive q ($q < M$) narrowband sources impinging from the directions $\theta_1, \ldots, \theta_q$. Assume that there are only N snapshots $\boldsymbol{x}(1), \boldsymbol{x}(2), \ldots, \boldsymbol{x}(N)$ available. The observation vectors can be modeled as

$$\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{n}(t)\ , \quad t = 1, 2, \ldots, N \tag{1}$$

where

$$\boldsymbol{A} = [\boldsymbol{a}(\theta_1), \ldots, \boldsymbol{a}(\theta_q)] \tag{2}$$

is the $M \times q$ matrix of signal wavefronts, $\boldsymbol{a}(\theta)$ is the $M \times 1$ steering vector, $\boldsymbol{s}(t)$ is the $q \times 1$ vector of source waveforms, and $\boldsymbol{n}(t)$ is the $M \times 1$ vector of sensor noise. Hence, the measured array data matrix

$$\boldsymbol{X} = [\boldsymbol{x}(1), \boldsymbol{x}(2), \ldots, \boldsymbol{x}(N)] \tag{3}$$

can be modeled as

$$\boldsymbol{X} = \boldsymbol{A}\boldsymbol{S} + \boldsymbol{N} \tag{4}$$

where

$$\boldsymbol{S} = [\boldsymbol{s}(1), \boldsymbol{s}(2), \ldots, \boldsymbol{s}(N)] \tag{5}$$

is the $q \times N$ matrix of source waveforms, and

$$\boldsymbol{N} = [\boldsymbol{n}(1), \boldsymbol{n}(2), \ldots, \boldsymbol{n}(N)] \tag{6}$$

is the $M \times N$ matrix of sensor noise.

3. UNITARY ESPRIT

According to [1], introduce the real-valued data matrix

$$\mathcal{T}(\boldsymbol{X}) = \boldsymbol{Q}_M^H [\boldsymbol{X} \quad \boldsymbol{\Pi}_M \boldsymbol{X}^* \boldsymbol{\Pi}_N]\, \boldsymbol{Q}_{2N} \tag{7}$$

where $\boldsymbol{\Pi}_M$ is the $M \times M$ matrix with ones on its antidiagonal and zeros elsewhere,

$$\boldsymbol{Q}_{2l} = \frac{1}{\sqrt{2}} \begin{bmatrix} \boldsymbol{I}_l & j\boldsymbol{I}_l \\ \boldsymbol{\Pi}_l & -j\boldsymbol{\Pi}_l \end{bmatrix} \tag{8}$$

$$\boldsymbol{Q}_{2l+1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \boldsymbol{I}_l & \boldsymbol{0} & j\boldsymbol{I}_l \\ \boldsymbol{0}^T & \sqrt{2} & \boldsymbol{0}^T \\ \boldsymbol{\Pi}_l & \boldsymbol{0} & -j\boldsymbol{\Pi}_l \end{bmatrix} \tag{9}$$

are, for example, the sparse unitary matrices defined in [1], $\boldsymbol{I}_l$ is the $l \times l$ identity matrix, $(\cdot)^H$ and $(\cdot)^*$ stand for Hermitian transpose and complex conjugate, respectively. The $M \times M$ sample covariance of the real-valued data matrix (7) is given by

$$\hat{\boldsymbol{R}}_x = \frac{1}{2N} \mathcal{T}(\boldsymbol{X}) \mathcal{T}(\boldsymbol{X})^H \tag{10}$$

Write the eigendecomposition of $\hat{\boldsymbol{R}}_x$ as

$$\hat{\boldsymbol{R}}_x = \boldsymbol{E}_{\rm S}\boldsymbol{\Lambda}_{\rm S}\boldsymbol{E}_{\rm S}^H + \boldsymbol{E}_{\rm N}\boldsymbol{\Lambda}_{\rm N}\boldsymbol{E}_{\rm N}^H \tag{11}$$

where $q \times q$ and $(M-q)\times(M-q)$ diagonal matrices $\boldsymbol{\Lambda}_{\rm S}$ and $\boldsymbol{\Lambda}_{\rm N}$ contain the q and $M-q$ sample signal and noise subspace eigenvalues, respectively, whereas the columns of the $M \times q$ and $M \times (M-q)$ matrices $\boldsymbol{E}_{\rm S}$ and $\boldsymbol{E}_{\rm N}$ contain the sample signal and noise subspace eigenvectors, respectively.

The Unitary ESPRIT algorithm is based on the solution of the real-valued invariance equation [1]

$$\underbrace{\boldsymbol{K}_1\boldsymbol{E}_{\rm S}}_{\mathbb{R}^{m\times q}}\boldsymbol{\Upsilon} \approx \underbrace{\boldsymbol{K}_2\boldsymbol{E}_{\rm S}}_{\mathbb{R}^{m\times q}} \tag{12}$$

by means of Least Squares (LS), Total LS (TLS), or Structured LS (SLS) [2]. Here, the $m \times M$ matrices $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are given by

$$\boldsymbol{K}_1 = \boldsymbol{Q}_m^H(\boldsymbol{J}_1+\boldsymbol{J}_2)\boldsymbol{Q}_M = 2\,\mathrm{Re}\{\boldsymbol{Q}_m^H\boldsymbol{J}_2\boldsymbol{Q}_M\} \tag{13}$$

$$\boldsymbol{K}_2 = \boldsymbol{Q}_m^H(\boldsymbol{J}_1-\boldsymbol{J}_2)\boldsymbol{Q}_M = 2\,\mathrm{Im}\{\boldsymbol{Q}_m^H\boldsymbol{J}_2\boldsymbol{Q}_M\} \tag{14}$$

where the $m \times M$ matrices $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ select the first and the last m ($m < M$) rows of an arbitrary matrix with the vertical dimension M [1], respectively. Write the eigendecomposition of the obtained real-valued $q \times q$ matrix $\boldsymbol{\Upsilon}$ as

$$\boldsymbol{\Upsilon} = \boldsymbol{T}\boldsymbol{\Omega}\boldsymbol{T}^{-1} \tag{15}$$

where $\boldsymbol{\Omega}$ is the $q \times q$ diagonal matrix of eigenvalues

$$\boldsymbol{\Omega} = \mathrm{diag}\{\omega_1, \omega_2, \ldots, \omega_q\} \tag{16}$$

and $\boldsymbol{T}$ is the $q \times q$ matrix of eigenvectors. Since $\boldsymbol{\Upsilon}$ is a real-valued matrix, it can happen that either all eigenvalues in (16) are real or some of them appear as complex conjugate pairs. The latter case corresponds to the unreliable DOA estimate when the associated signal sources are not resolved (i.e., they merge and therefore result into a complex conjugate eigenvalue pair). In [1], it is proposed to exploit these eigenvalue properties in a reliability test:

Check whether all eigenvalues ω_i, $i = 1, 2, \ldots, q$ of $\boldsymbol{\Upsilon}$ in (15) are real-valued.

If this test fails, it is recommended in [1] and [2] to start Unitary ESPRIT again with more reliable measurements, or to use more snapshots. However, in many practical situations neither more reliable measurements nor additional data snapshots are available. The idea of our approach presented below is to exploit synthetically generated (resampled) data instead of unavailable additional measured data.

If the reliability test is satisfied in Unitary ESPRIT, the resulting signal DOA's can be estimated in a straightforward manner. For example, for a ULA with maximum overlap [1], the estimates of the signal DOA's are obtained as

$$\hat{\theta}_i = \arcsin\left(\frac{\lambda}{\pi d}\arctan(\omega_i)\right), \quad i = 1, 2, \ldots, q \tag{17}$$

where d is the interelement spacing, and λ is the wavelength. If the reliability test fails, it is meaningful to omit the complex parts of the complex conjugate eigenvalues ω_i. Therefore, the estimates of the signal DOA's are then given by

$$\hat{\theta}_i = \arcsin\left(\frac{\lambda}{\pi d}\arctan(\mathrm{Re}\{\omega_i\})\right), \quad i = 1, 2, \ldots, q \tag{18}$$

Note that in this situation the outlier occurs, because each pair of unresolved signals is attributed to a single real part of a complex conjugate eigenvalue pair.

4. PSEUDO-NOISE RESAMPLING

Pseudo-noise-based techniques are known in a variety of applications (e.g., see [5] and [6]). In case of a failed reliability test, the aim of our pseudo-noise resampling technique is to remove the outlier and to recover the Unitary ESPRIT performance using *exactly the same data snapshots* that have been used in Unitary ESPRIT. The central idea is to resample the data matrix several times using synthetically generated pseudo-noise [4], to run Unitary ESPRIT "in parallel" for each such resampling, and then to select only the "successful" runs for which the reliability test is satisfied and the outlier is removed. The $M \times N$ resampled data matrix is given by

$$\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{Z} \tag{19}$$

where $\boldsymbol{Z}$ is the $M \times N$ matrix of independent zero-mean circular pseudo-noise drawn from a random generator:

$$\mathrm{E}\{\boldsymbol{Z}\} = \boldsymbol{0}\,, \quad \mathrm{E}\{\boldsymbol{Z}\boldsymbol{Z}^H\} = \sigma_Z^2 N\boldsymbol{I}\,, \quad \mathrm{E}\{\boldsymbol{Z}\boldsymbol{Z}^T\} = \boldsymbol{0} \tag{20}$$

Repeating the resampling runs (19), we can obtain more and more "synthetically" generated data matrices.

In order to maintain an acceptable Signal to Noise Ratio (SNR) in the synthetic (resampled) data, the variance of the pseudo-noise σ_Z^2 should have approximately the same order as the variance σ^2 of the original noise (i.e, σ_Z^2 should not be too high). A similar constraint is exploited in [6]. The consistent estimate of σ^2 is given by [7]

$$\hat{\sigma}^2 = \frac{1}{M-q}\,\mathrm{trace}\{\boldsymbol{\Lambda}_{\rm N}\} = \frac{1}{M-q}\sum_{i=q+1}^{M}\lambda_i \tag{21}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M$ are the ordered eigenvalues of the sample covariance matrix $\hat{\boldsymbol{R}}_x$. In practice, it is meaningful to determine the variance of the pseudo-noise as

$$\sigma_Z^2 = p \cdot \hat{\sigma}^2 \tag{22}$$

where $p \sim 1$ is a constant chosen by the user.

Motivated by the success of modern resampling schemes (e.g., bootstrap and jackknife), it is our goal that the pseudo-noise $\boldsymbol{Z}$ will redistribute the original noise $\boldsymbol{N}$ between array sensors in a favorable way. Thereby, an improved estimation performance can be achieved in the successful resampling runs (i.e., where the reliability test is passed). In this approach, the reliability test can be interpreted as a censored selector of the results of the pseudo-random resampling.

We stress that the synthetically generated noise $\boldsymbol{Z}$ always decreases the SNR, and therefore *the resampling itself cannot bring any performance improvement*. However, *it is no longer true* if we use the reliability test (censoring) to select only outlier-free estimators from the whole number of resampled estimators.

5. THE PROPOSED TECHNIQUE

Let us now formulate the new algorithm which employs the synthetic (resampled) data (19) every time the reliability test has failed for the measured data (3). Assume that after K resampling runs we obtain K "parallel" Unitary ESPRIT estimators, where each of them is applied to a different resampled data matrix (19). Let the ith estimator be

$$\boldsymbol{\theta}^{(i)} = \{\hat{\theta}_l^{(i)}\}_{l=1}^{q} \tag{23}$$

where $\hat{\theta}_1^{(i)} \leq \hat{\theta}_2^{(i)} \leq \cdots \leq \hat{\theta}_q^{(i)}$ is the ordered set of Unitary ESPRIT DOA estimates corresponding to the ith resampling run. Then, provided that the data matrix $\boldsymbol{X}$ is fixed, these estimators are said to form the *estimator bank* [3]

$$\mathcal{F} = \{\boldsymbol{\theta}^{(i)},\ i = 1, 2, \ldots, K\} \tag{24}$$

of dimension K.

Divide (24) in two disjoint subsets

$$\mathcal{F}_1 = \{\tilde{\boldsymbol{\theta}}^{(i)},\ i = 1, 2, \ldots, J\} \tag{25}$$

$$\mathcal{F}_2 = \{\overline{\boldsymbol{\theta}}^{(i)},\ i = 1, 2, \ldots, K - J\} \tag{26}$$

where the first subset $\mathcal{F}_1$ contains J estimators that pass the reliability test, whereas the second subset $\mathcal{F}_2$ contains the remaining $K - J$ estimators for which this test fails.

Table 1: Summary of the proposed algorithm

1. **Compute the standard Unitary ESPRIT DOA estimator using the measured data matrix (3)**.
2. **Apply the reliability test to this estimator**.
   - If the reliability test has no failure then estimate the signal DOA's using (17) and go to step 4.
   - If the reliability test fails then form the estimator bank (24) using multiple pseudo-noise resamplings (19) of the measured data matrix (3).
3. **Apply the reliability test to each resampled estimator from the estimator bank (24)**.
   - If any J ($J > 0$) estimators from (24) pass the reliability test then estimate the signal DOA's using (28).
   - If the reliability test fails for all estimators from (24) (i.e., $J = 0$) then use standard Unitary ESPRIT and estimate the signal DOA's using (18).
4. **Stop the algorithm**.

Consider the case when the first subset contains at least one estimator, i.e., let $0 < J \leq K$. Let the estimators in the first subset be given by

$$\tilde{\boldsymbol{\theta}}^{(i)} = \{\tilde{\theta}_l^{(i)}\}_{l=1}^{q}\ , \quad i = 1, 2, \ldots, J \tag{27}$$

Apparently, an appropriate combination of the results of these "successfully" resampled estimators is necessary to obtain a final estimate. Assuming that the DOA's in (27) are sorted as $\tilde{\theta}_1^{(i)} < \tilde{\theta}_2^{(i)} < \cdots < \tilde{\theta}_q^{(i)}$ for all $i = 1, 2, \ldots, J$, exploit the robust median averager to obtain the final DOA estimates [3]

$$\hat{\theta}_l = \text{med}\,\{\tilde{\theta}_l^{(1)}, \tilde{\theta}_l^{(2)}, \ldots, \tilde{\theta}_l^{(J)}\}\ , \quad l = 1, 2, \ldots, q \tag{28}$$

where for arbitrary real $b_1, b_2 \ldots, b_J$

$$\text{med}\,\{b_1, \ldots, b_J\} = \begin{cases} (c_{\frac{J}{2}} + c_{\frac{J}{2}+1})/2, & \text{if } J \text{ is even} \\ c_{\frac{J+1}{2}}, & \text{if } J \text{ is odd} \end{cases} \tag{29}$$

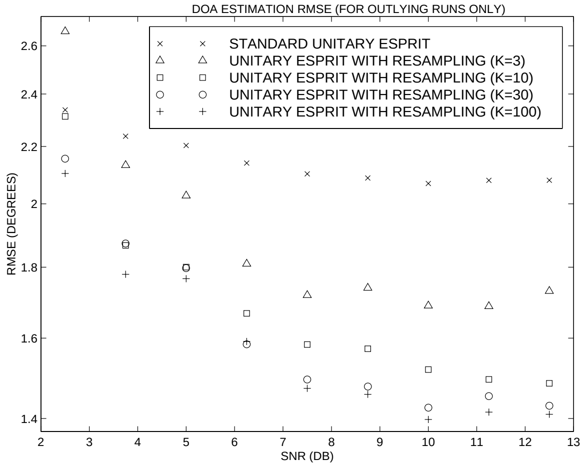

Figure 1: RMSE's vs. the SNR in the first example.

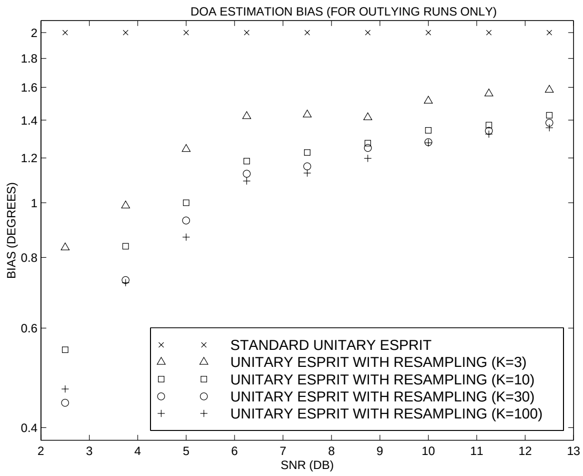

Figure 2: Biases vs. the SNR in the first example.

$$\{c_1, \ldots, c_J\} = \text{sort}\,\{b_1, \ldots, b_J\} \tag{30}$$

and $\text{sort}\,\{\cdot\}$ stands for the operator of sorting in ascending (descending) order. The proposed algorithm is based on equations (17), (18), (28) and is summarized in Table 1.

It should be noted that our algorithm requires more computations than Unitary ESPRIT by a factor $1 + K r_{\text{out}}$ where $0 \leq r_{\text{out}} \leq 1$ is the reliability test failure rate for Unitary ESPRIT. Note that according to our simulations, typical values of the parameter r_{out} are concentrated approximately between 0 and 0.15. Taking into account the low computational complexity of Unitary ESPRIT, it can be concluded that for a moderate K the proposed algorithm enables an efficient implementation, which can be parallelized easily. Apparently, this additional increase of the computational burden relative to the Unitary ESPRIT algorithm can be viewed as a natural payment for the improved (outlier-free) performance.

6. SIMULATION RESULTS

In all simulations to follow, we assume a ULA of $M = 6$ omnidirectional sensors with a half-wavelength spacing, $N = 100$

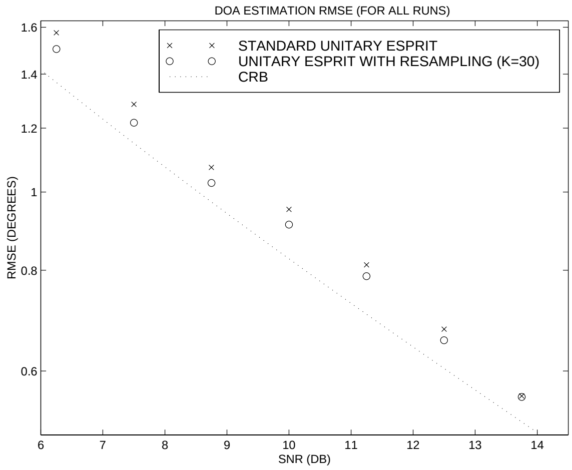

Figure 3: RMSE's vs. the SNR in the first example.

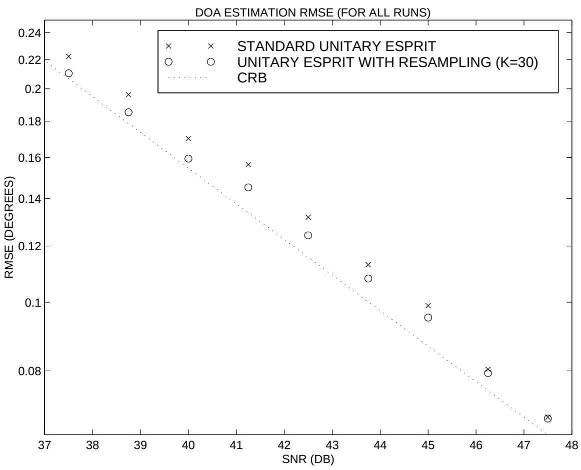

Figure 4: RMSE's vs. the SNR in the second example.

snapshots, uncorrelated equipower sources with zero-mean Gaussian waveforms, and zero-mean white Gaussian noise. The pseudo-noise parameter $p = 0.2$ was taken in (22) and all results were averaged over 1000 simulation runs. Throughout the simulations, LS Unitary ESPRIT was used, motivated by its better performance and lower computational burden relative to TLS Unitary ESPRIT [2].

In the first example, we assumed three sources impinging from $\theta_1 = 0°$, $\theta_2 = 4°$, and $\theta_3 = 30°$. Fig. 1 shows the DOA estimation Root Mean Square Errors (RMSE's) of Unitary ESPRIT and the proposed algorithm (with four fixed values of K) versus the SNR. Note that all curves in this figure were averaged *only over the simulation runs which correspond to outlying Unitary ESPRIT estimates*. These curves were additionally averaged over the first two sources. Similar curves for the DOA estimation bias are displayed in Fig. 2. From these figures we observe that the restored outlier-free estimates are biased and the bias does not decrease with increasing SNR. This bias can be interpreted as a natural payment for resolving closely spaced sources. However, the RMSE (which includes the bias component, too) tends to decrease when the SNR and parameter K grow. Note that both the bias and the RMSE are significantly lower after the resampling than that in outlying Unitary ESPRIT estimates, and this proves the positive effect of resampling. From Figs. 1 and 2 one can conclude that the value $K = 30$ is sufficient to obtain a satisfactory performance.

As the outlier rate may change with SNR, it is interesting to study how the statistical performance is improved when the averaging is done over all simulation runs, irrespective of the Unitary ESPRIT performance in each run. Fig. 3 shows the RMSE's of Unitary ESPRIT and the proposed algorithm (with $K = 30$) versus the SNR for the first example. In contrast to Fig. 1, the results in this figure are averaged over *all simulation runs* as well as over the first two sources.

In the second example, we assumed two sources with $\theta_1 = 0°$ and $\theta_2 = 0.5°$. The estimation RMSE's of Unitary ESPRIT and the proposed algorithm (with $K = 30$) versus the SNR are plotted for this example in Fig. 4. Again, the results were averaged over all simulation runs and both sources.

From Figs. 3 and 4 we can observe noticeable statistical performance improvements achieved via the proposed algorithm relative to standard Unitary ESPRIT. These improvements are more pronounced in the specific "transition" region between the so-called threshold and asymptotic domains, where outliers affect the statistical performance of Unitary ESPRIT significantly.

7. CONCLUSIONS

The proposed resampling approach has been shown to enable the mitigation of outliers in Unitary ESPRIT. Our technique results in an improved statistical performance in the "transition" region, where the outliers have a strong contribution to the statistical performance of Unitary ESPRIT. The proposed algorithm is free of typical limitations of other known resampling (estimator bank) techniques in that it does not require any knowledge on signal localization sectors.

8. REFERENCES

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