BOUNDARY FILTERS WITHOUT DC LEAKAGE FOR PARAUNITARY FILTER BANKS

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ABSTRACT

The construction of boundary filters without DC leakage for two-channel paraunitary FIR filter banks is considered. The design procedure is based on orthogonal boundary filters which are optimal in a weighted mean square error sense in the Fourier domain and on Householder transformation of boundary filter matrices. Simulation results are presented for boundary filters based on minimum-phase Daubechies filters.

1. INTRODUCTION

Two-channel perfect reconstruction filter banks are widely used in subband coding applications. For infinite-length signals, theory and design of two-channel filter banks are mature and well documented subjects of digital signal processing [1], [2]. However, in a variety of applications, processing of finite-length signals is required. Image processing and image coding are prime examples. For perfect reconstruction filter banks with linear-phase analysis and synthesis filters the symmetric extension method is an established solution [3], [4], [5]. Since symmetry and orthogonality of the prototype filters for two-channel paraunitary FIR filter banks are incompatible, circular extension is often applied. However, discontinuities at the signal boundaries introduce artificial high frequency components and degrade coding performance [6]. Alternative signal extension methods, e.g., zero-padding or edge-value replication lead to distortion at the signal boundaries or introduce redundancy [6].

In [7] and [8] orthogonal time-varying filter banks were introduced. Orthogonal boundary filters were constructed to apply paraunitary FIR filter banks to one-sided and finite-length signals without introducing redundancy or distortion. Time-varying filter banks were also constructed in [9] but orthogonality of the boundary regions was not preserved. Optimal orthogonal boundary filter banks with respect to coding gain were presented in [10]. The singular value decomposition (SVD) was employed for construction of optimal boundary filters for given second-order statistics in [11]. In [12] and [13] orthogonal boundary filters which are optimal in the weighted mean square error (MSE) sense in the Fourier domain were constructed based on a modification of the orthogonal Procrustes problem. Simultaneous optimization of boundary filters and stationary filters was considered in [14].

The scope of this paper is the design of orthogonal boundary filters without DC leakage, i.e., the boundary highpass filters have at least a single zero at $z = 1$. This design constraint is of particular interest for coding applications since DC leakage significantly degrades coding performance. The design procedure is an extension of the method presented in [13] and employs Householder transformation of boundary filter matrices.

The outline of the paper is as follows. In section 2, construction of boundary filters for two-channel paraunitary FIR filter banks is reviewed. The design of boundary filters which are optimal in the weighted MSE sense in the Fourier domain is outlined in section 3. In section 4, the proposed design procedure for boundary filters without DC leakage is presented.

The following notation is used in the paper. Boldfaced quantities denote matrices or column vectors. Row vectors are denoted as transposed column vectors. The row and column indices of matrices and vectors are counted from zero. The quantities $\boldsymbol{A}'$, $\boldsymbol{A}''$ and $\boldsymbol{A}^{\mathrm{T}}$ denote the real part, the imaginary part, and the transpose of $\boldsymbol{A}$, respectively. $\boldsymbol{O}_{N\times M}$ denotes the $N \times M$ zero matrix and $\boldsymbol{I}_N$ denotes the $N \times N$ identity matrix.

2. CONSTRUCTION OF BOUNDARY FILTERS

For the sequel, let $\boldsymbol{h}_0 = [h(N-1), h(N-2), ..., h(0)]^{\mathrm{T}} \in \mathbb{R}^N$ denote the reversed lowpass analysis filter impulse response vector of a causal two-channel paraunitary FIR filter bank. The corresponding reversed highpass analysis filter impulse response vector is given by $\boldsymbol{h}_1 = [h(0), -h(1), ..., -h(N-1)]^{\mathrm{T}}$ [15]. Note that $N = 2M$, $h(N-1) \neq 0$, and $\boldsymbol{h}_0^{\mathrm{T}}\boldsymbol{h}_0 = \boldsymbol{h}_1^{\mathrm{T}}\boldsymbol{h}_1 = 1$ hold. For construction of boundary filters the $(N-2) \times (N-2)$ triangular block matrices

$$\boldsymbol{A}_0 = \begin{bmatrix} \boldsymbol{H}_0 & \boldsymbol{H}_1 & \cdots & \boldsymbol{H}_{M-2} \\ & \boldsymbol{H}_0 & \cdots & \boldsymbol{H}_{M-3} \\ & & & \cdots \\ & & & \boldsymbol{H}_0 \end{bmatrix} \tag{1}$$

$$\boldsymbol{A}_1 = \begin{bmatrix} \boldsymbol{H}_{M-1} & & & \\ \boldsymbol{H}_{M-2} & \boldsymbol{H}_{M-1} & & \\ \cdots & & & \\ \boldsymbol{H}_1 & \boldsymbol{H}_2 & \cdots & \boldsymbol{H}_{M-1} \end{bmatrix} \tag{2}$$

with $M = N/2$ and

$$\boldsymbol{H}_m = \begin{bmatrix} h(N-1-2m) & h(N-2-2m) \\ h(2m) & -h(2m+1) \end{bmatrix} \tag{3}$$

are employed. From [8] it is known that there exist exactly $M-1$ left and right boundary filters each of maximal length $N-2$. Canonical sets of boundary filters can be obtained by downsampling the rows of $\boldsymbol{A}_0$ and $\boldsymbol{A}_1$, respectively. Let $\boldsymbol{a}_0^{\mathrm{T}}, ..., \boldsymbol{a}_{N-3}^{\mathrm{T}}$ denote the $N-2$ row vectors of $\boldsymbol{A}_1$. Then, the $M-1$ row vectors $\boldsymbol{a}_1^{\mathrm{T}}, \boldsymbol{a}_3^{\mathrm{T}}, ..., \boldsymbol{a}_{N-3}^{\mathrm{T}}$ are linearly independent and orthogonal to the row space of $\boldsymbol{A}_0$ [13]. Hence, Gram-Schmidt orthogonalization and subsequent orthonormalization of $\boldsymbol{a}_1^{\mathrm{T}}, \boldsymbol{a}_3^{\mathrm{T}}, ..., \boldsymbol{a}_{N-3}^{\mathrm{T}}$ yields a canonical set of left boundary filters. Correspondingly, let $\boldsymbol{b}_0^{\mathrm{T}}, ..., \boldsymbol{b}_{N-3}^{\mathrm{T}}$ denote the $N-2$ row vectors of $\boldsymbol{A}_0$. Then, the $M-1$ row vectors $\boldsymbol{b}_0^{\mathrm{T}}, \boldsymbol{b}_2^{\mathrm{T}}, ..., \boldsymbol{b}_{N-4}^{\mathrm{T}}$ are linearly independent and orthogonal to the row space of $\boldsymbol{A}_1$ [13] and a canonical set of right boundary filters can be obtained by Gram-Schmidt orthogonalization and subsequent orthonormalization of $\boldsymbol{b}_0^{\mathrm{T}}, \boldsymbol{b}_2^{\mathrm{T}}, ..., \boldsymbol{b}_{N-4}^{\mathrm{T}}$.

3. OPTIMIZATION OF BOUNDARY FILTERS

If $\boldsymbol{R}_0$ is a left boundary filter matrix then

$$\boldsymbol{B}_0 = \boldsymbol{Q}_0^{\mathrm{T}} \boldsymbol{P}_0 = \boldsymbol{Q}_0^{\mathrm{T}} \begin{bmatrix} \boldsymbol{I}_{p_0} & \boldsymbol{O}_{p_0 \times (N-2)} \\ \boldsymbol{O}_{(M-1)\times p_0} & \boldsymbol{R}_0 \end{bmatrix} \tag{4}$$

is also a left boundary matrix where $\boldsymbol{Q}_0$ denotes an $(M-1+p_0) \times (M-1+p_0)$ orthogonal matrix [8]. Similarly, if $\boldsymbol{R}_1$ is a right boundary filter matrix then

$$\boldsymbol{B}_1 = \boldsymbol{Q}_1^{\mathrm{T}} \boldsymbol{P}_1 = \boldsymbol{Q}_1^{\mathrm{T}} \begin{bmatrix} \boldsymbol{R}_1 & \boldsymbol{O}_{(M-1)\times p_1} \\ \boldsymbol{O}_{p_1 \times (N-2)} & \boldsymbol{I}_{p_1} \end{bmatrix} \tag{5}$$

is also a right boundary filter matrix where $\boldsymbol{Q}_1$ denotes an $(M-1+p_1) \times (M-1+p_1)$ orthogonal matrix [8]. Note that there are $M-1+p_0$ left boundary filters each of maximal length $N-2+p_0$ and $M-1+p_1$ right boundary filters each of maximal length $N-2+p_1$. The objective function for optimization of the boundary filters will be the weighted MSE between the frequency responses of the boundary filters and the corresponding stationary filters. If the numbers of boundary filters $M-1+p_0$ and $M-1+p_1$ are restricted to be even, the corresponding stationary filter matrices are

$$\boldsymbol{S}_i = \begin{bmatrix} \boldsymbol{H}_0 & \boldsymbol{H}_1 & \cdots & \boldsymbol{H}_{M-1} \\ \cdots & & & \\ \boldsymbol{H}_0 & \cdots & & \boldsymbol{H}_{M-1} \end{bmatrix} \tag{6}$$

where $\boldsymbol{S}_i \in \mathbb{R}^{(M-1+p_i)\times N}$, $i = 0, 1$ holds. With the Fourier matrices $\boldsymbol{F}_{K\times L} = [W^{kl}]_{0\le k \le K-1,\, 0 \le l \le L-1}$, $\boldsymbol{F} = \boldsymbol{F}_{N\times L}$, $\boldsymbol{F}_i = \boldsymbol{F}_{(N-2+p_i)\times L}$, $W = \exp(-j2\pi/L)$, and the positive definite frequency weight matrix $\boldsymbol{W} = \mathrm{diag}(w_0, ..., w_{L-1})$, the boundary filter optimization problem can be stated as

$$\min_{\boldsymbol{Q}_i} \|\boldsymbol{S}_i \boldsymbol{F} \boldsymbol{W} - \boldsymbol{Q}_i^{\mathrm{T}} \boldsymbol{P}_i \boldsymbol{F}_i \boldsymbol{W}\|_F \tag{7}$$

subject to the constraint $\boldsymbol{Q}_i^{\mathrm{T}} \boldsymbol{Q}_i = \boldsymbol{I}_{M-1+p_i}$. Given $\boldsymbol{T}_i = \boldsymbol{T}_i' + j\,\boldsymbol{T}_i'' \in \mathbb{C}^{K\times L}$, an orthogonal matrix $\boldsymbol{Q} \in \mathbb{R}^{K\times K}$ which minimizes

$$\|\boldsymbol{T}_1 - \boldsymbol{Q}^{\mathrm{T}} \boldsymbol{T}_2\|_F \tag{8}$$

is $\boldsymbol{Q} = \boldsymbol{U}\boldsymbol{V}^{\mathrm{T}}$ [13] with the SVD

$$\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{T}} = \boldsymbol{T}_2' {\boldsymbol{T}_1'}^{\mathrm{T}} + \boldsymbol{T}_2'' {\boldsymbol{T}_1''}^{\mathrm{T}}. \tag{9}$$

The optimal boundary filter matrices $\boldsymbol{B}_i = \boldsymbol{Q}_i^{\mathrm{T}} \boldsymbol{P}_i$ are now readily obtained by setting $\boldsymbol{T}_1 = \boldsymbol{S}_i \boldsymbol{F} \boldsymbol{W}$ and $\boldsymbol{T}_2 = \boldsymbol{P}_i \boldsymbol{F}_i \boldsymbol{W}$, respectively.

With the decompositions $\boldsymbol{B}_0 = [\boldsymbol{B}_{00} \ \ \boldsymbol{B}_{01}]$ and $\boldsymbol{B}_1 = [\boldsymbol{B}_{11} \ \ \boldsymbol{B}_{10}]$, $\boldsymbol{B}_{i0} \in \mathbb{R}^{(M-1+p_i)\times p_i}$, $\boldsymbol{B}_{i1} \in \mathbb{R}^{(M-1+p_i)\times(N-2)}$, the smallest possible orthogonal matrix is

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{B}_{00} & \boldsymbol{B}_{01} & & \\ & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \\ & & \boldsymbol{B}_{11} & \boldsymbol{B}_{10} \end{bmatrix}. \tag{10}$$

Hence, the minimal required signal length is $L_0 = 2(N-2) + p_0 + p_1$. For signal lengths $L > L_0$, the corresponding subband decomposition matrix can be readily obtained by straightforward extension of the block matrix $[\boldsymbol{A}_0 \ \boldsymbol{A}_1]$.

Fig. 1 shows optimal length-3 boundary filters based on minimum-phase length-4 Daubechies filters [2], i.e., $p_0 = p_1 = 1$. Note that the highpass boundary filters reveal considerable DC leakage. Fig. 2 illustrates orthogonal subband decomposition of a constant length-16 signal. Note that the first and last sample of the decimated highpass signal are not equal to zero as a consequence of DC leakage of the highpass boundary filters.

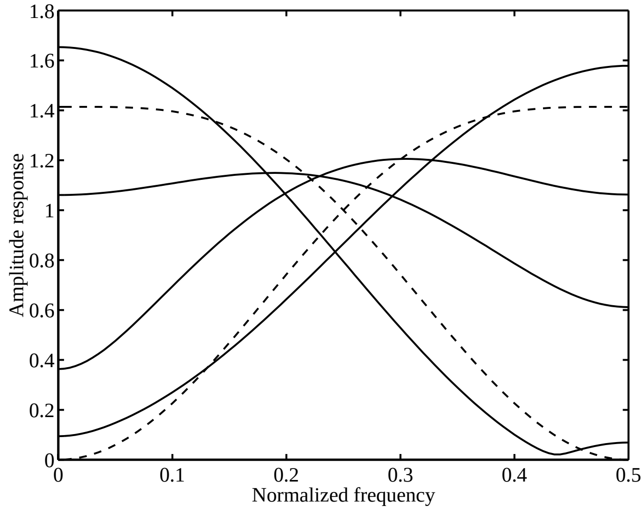

Figure 1: Optimal length-3 boundary filters (solid lines) based on minimum-phase length-4 Daubechies filters (dashed lines). The highpass boundary filters reveal considerable DC leakage.

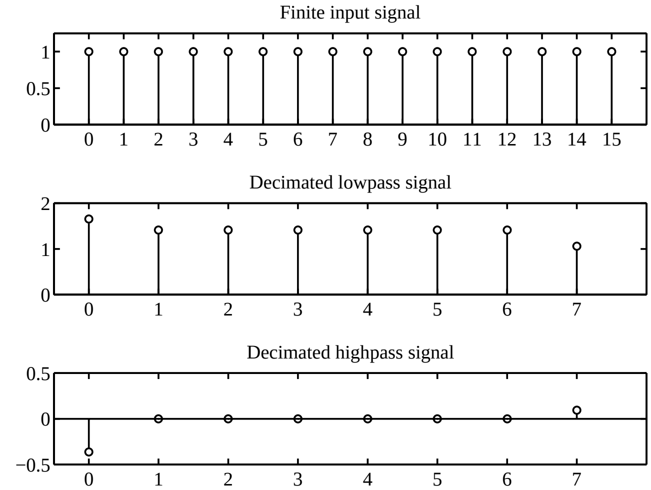

Figure 2: Orthogonal subband decomposition of a constant length-16 signal using minimum-phase length-4 Daubechies filters along with optimal length-3 boundary filters.

4. BOUNDARY FILTERS WITHOUT DC LEAKAGE

For construction of boundary filters without DC leakage, the matrices $\boldsymbol{Q}_i^{\mathrm{T}}$ in (4) and (5) are rewritten as the product of Householder matrices $\boldsymbol{U}_i^{\mathrm{T}} = \boldsymbol{U}_i$ and general orthogonal matrices $\boldsymbol{V}_i^{\mathrm{T}}$, i.e.,

$$\boldsymbol{Q}_i^{\mathrm{T}} = \boldsymbol{U}_i^{\mathrm{T}} \boldsymbol{V}_i^{\mathrm{T}}. \tag{11}$$

The matrices $\boldsymbol{V}_i$ are determined according to the solution of the optimal design procedure (7). The Householder matrices [16]

$$\boldsymbol{U}_i = \boldsymbol{I} - 2\boldsymbol{u}_i \boldsymbol{u}_i^{\mathrm{T}} \tag{12}$$

are determined to satisfy the requirement

$$\boldsymbol{S}_i \boldsymbol{f} = \boldsymbol{U}_i^{\mathrm{T}} \boldsymbol{V}_i^{\mathrm{T}} \boldsymbol{P}_i \boldsymbol{f}_i \tag{13}$$

with $\boldsymbol{f} = \boldsymbol{F}_{N\times 1}$ and $\boldsymbol{f}_i = \boldsymbol{F}_{(N-2+p_i)\times 1}$. Note that $\boldsymbol{f}_i = [1,1,...,1]^{\mathrm{T}}$ and $\boldsymbol{s}_i = \boldsymbol{S}_i \boldsymbol{f} = \sqrt{2}\,[1,0,...,1,0]^{\mathrm{T}}$ hold. This time domain constraint ensures at least a single zero of the boundary highpass filters at $z = 1$. With

$$\boldsymbol{b}_i = \boldsymbol{V}_i^{\mathrm{T}} \boldsymbol{P}_i \boldsymbol{f}_i \tag{14}$$

the unit-norm vectors $\boldsymbol{u}_i$ in (12) can be chosen to

$$\boldsymbol{u}_i = \frac{\boldsymbol{s}_i - \boldsymbol{b}_i}{\|\boldsymbol{s}_i - \boldsymbol{b}_i\|_2}. \tag{15}$$

Summarizing, boundary filter matrices without DC leakage are given by

$$\boldsymbol{B}_i = (\boldsymbol{I} - 2\boldsymbol{u}_i \boldsymbol{u}_i^{\mathrm{T}}) \boldsymbol{V}_i^{\mathrm{T}} \boldsymbol{P}_i. \tag{16}$$

Fig. 3 shows length-3 boundary filters without DC leakage based on minimum-phase length-4 Daubechies filters. Interestingly, these filters are identically equal to the boundary filters found by numerical means in [17] and applied for construction of tree-structured signal expansions in [18]. Fig. 4 illustrates orthogonal subband decomposition of a constant length-16 signal. Note that the decimated highpass signal vanishes whereas the decimated lowpass signal remains almost unchanged compared to Fig. 2.

5. REFERENCES

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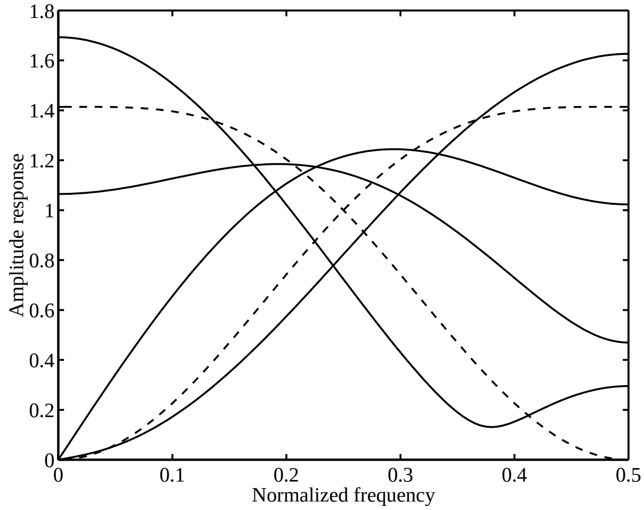

Figure 3: Length-3 boundary filters (solid lines) without DC leakage based on minimum-phase length-4 Daubechies filters (dashed lines).

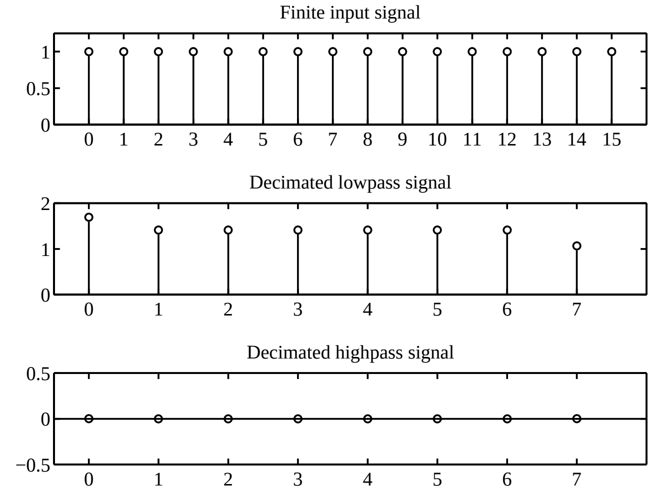

Figure 4: Orthogonal subband decomposition of a constant length-16 signal using minimum-phase length-4 Daubechies filters along with length-3 boundary filters without DC leakage.

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