FROM GAUSSIAN SCALE-SPACE TO B-SPLINE SCALE-SPACE

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ABSTRACT

The Gaussian kernel has long been used in the classical multiscale analysis. The purpose of the paper is to propose the uniform B-spline as an alternative for the visual modeling. A general framework for various scale-space representations is formulated using the B-spline approach. In particular, the evolution of the wavelet models can be well understood from such an approach. Most of the wavelet representations can be factored into B-spline bases and hence can be implemented efficiently using the spline technique. Besides, it is shown that the B-spline scale-space representations not only inherit most of the properties of the Gaussian scale-space but also have many advantages with respect to the efficiency, compactness and parallel structure.

1. INTRODUCTION

The multiscale representations of an image are of crucial importance if one aims at describing the structure of the image [5]. Usually the formation of the image is regarded as the heat diffusion process and hence the Gaussian kernel is widely used in the classical multiscale analysis. More generally, the multiscale geometry of the image can be described by the non-linear PDEs. The reason that the Gaussian kernel is popularly used in the classical scale-space theory is twofold. It was proved that the Gaussian is the unique kernel which satisfies the causality property as guaranteed by the scaling theorem [12]. Moreover, the neurophysiological experiments have shown that the measured response profiles in the mammalian retina and visual cortex can be well-modeled by the superposition of Gaussian derivatives [13].

In practice, since the computational load becomes extremely heavy when the scale gets larger, B-spline based techniques have been used for the fast implementation of the scale-space filtering [6], [7], [8]. Moreover, due to the nice properties of the B-splines for multiscale analysis [3], most of the current wavelet models are derived from B-spline kernels. We will show that these wavelet models have the intrinsic relations with the the classical scale-space approach and can be realized more efficiently by using spline technique.

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2. B-SPLINE KERNELS AND THEIR PROPERTIES

The central continuous B-spline of order n is denoted by $\beta^n(x)$, which can be generated by repeated $n+1$ convolution of a B-spline of order 0,

$$\beta^n(x) = \beta^0 * \beta^{n-1}(x) = \overbrace{\beta^0 * \beta^0 * \cdots * \beta^0(x)}^{n+1}, \quad x \in R. \tag{1}$$

The discrete sampled B-spline $b_m^n(k)$ of order n and integer coarseness $m \geq 1$ is obtained by directly sampling the nth-order continuous B-spline at the scale m:

$$b_m^n(k) = \frac{1}{m}\beta^n(\frac{k}{m}) \quad \forall k \in Z. \tag{2}$$

The discrete B-spline of order n at scale m is defined as

$$B_m^n = \overbrace{B_m^0 * B_m^0 * \cdots B_m^0}^{n+1}, \tag{3}$$

where $B_m^0 = \frac{1}{m}[1, 1, \cdots, 1]$ is the sampled pulse of width m.

One significant property of the B-spline of a given order n is that it is the unique compactly supported refinable spline function of order n which can provide a stable hierarchical representation of a signal at different scales. It has been proven [3] that a compactly supported spline is m-refinable and stable if and only if it is a shifted B-spline. Let $h > 0$ and define the polynomial spline space S_h^n consisting of the dilated and shifted B-splines of order n (n is odd, which we will assume throughout the paper) by

$$S_h^n = \left\{ \sum_{k=-\infty}^{+\infty} c_m(k)\beta_h^n(x - hk) : c_m \in l^2 \right\}. \tag{4}$$

Then

$$S_{im}^n \subset S_m^n, \quad \forall i \in Z_+, \tag{5}$$

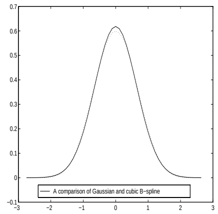

Figure 1: Good approximations of the cubic B-spline (in dotted line) to the Gaussian kernel (in solid line).

and

$$\overline{\bigcup_{h>0} S_h^n} = L^2(R). \tag{6}$$

The embedding property (5) follows from the fact that the B-spline $\beta^n(x)$ is m-refinable, i.e., it satisfies the following m-scale relation,

$$\frac{1}{m}\beta^n(\frac{x}{m}) = \sum_{k=-\infty}^{+\infty} B_m^n(k)\beta^n(x-k). \tag{7}$$

Another property is that B-splines are good approximations of the Gaussian kernel due to the central limit theorem. In [17] Unser, Aldroubi and Eden have presented a more general proof that B-splines converge to the Gaussian function in $L^p(R), \forall p \in [2,+\infty)$ as the order of the spline n tends to infinity. See Fig. 1 for illustration.

3. FAST IMPLEMENTATION OF CONTINUOUS SCALE-SPACE USING B-SPLINES

The traditional scale-space approach can be regarded as a continuous wavelet transform of the signal $f \in L^2$,

$$Wf(s,x) = \int f(t)\psi_s(t-x)dt \quad s>0 \tag{8}$$

where $\psi_s(x) = \frac{1}{s}\psi(\frac{x}{s}) \in L^2$ is the scaled wavelet.

Since spline spaces S_m^n provide close and stable approximations of $L^2(R)$, it is reasonable to approximate both the discrete signal and the wavelets using B-spline bases. We use the translated B-splines of order n_1 and n_2 as bases to approximate both the signal and the wavelets,

$$f(x) \approx \tilde{f}(x) = \sum_k c(k)\beta^{n_1}(x-k), \tag{9}$$

$$\psi(x) = \sum_k g(k)\beta^{n_2}(x-k). \tag{10}$$

Then using the m-refinable property of the B-splines, one can derive the cascaded filter bank algorithm for the fast implementation of the continuous wavelet transform at the rational scales [1]:

$$Wf(\frac{m_1}{m_2}, r) = m_2(b^{n_1+n_2+1} * B_{m_2}^{n_1} * B_{m_1}^{n_2} * c_{\uparrow m_2} * g_{\uparrow m_1})_{\downarrow m_2}(r). \tag{11}$$

where $\uparrow m$ and $\downarrow m$ denote the up-sampling and down-sampling operations by a factor m. The computational complexity of the above algorithm lies in the convolutions with $B_{m_2}^{n_1}$ and $B_{m_1}^{n_2}$. From (3) they can be factored as the repeated convolutions with the impulse of width m and then can be realized with only addition operations by the moving average technique with O(N) complexity. Such algorithm give an extension to the $\grave{a}$ trous algorithm [19] and the algorithm in [16] which can only compute the continuous wavelet transforms at the dyadic scales or integer scales. Finally, it is also obvious that such an algorithm can be realized parallelly for different scales.

4. DYADIC SCALE-SPACE FRAME REPRESENTATIONS

The derivatives of the Gaussian kernel such as the Marr-Hildreth operator [9] and the Canny operator have been widely used for the multiscale geometric analysis of the image. However, it has not been clear whether such representations are invertible. Using B-spline techniques efficient frame algorithms can be designed to express the image in terms of its mutiscale derivatives. These differential operators include the gradient operator, the second-derivative operator, the Laplacian operator and the multi-directional operator [2].

As an example, we show how to design a LoG-like or Mexican hat wavelets [9] using B-spline kernels. This representation is meaningful because it indicates that an image can be recovered from its multiscale LoG-like components. The scaling function is taken as the radial B-spline $\phi(x,y)$ which is a non-separable function of two variables defined by its Fourier transform

$$\hat{\phi}(\omega_x,\omega_y) = \hat{\beta}^n(\rho) \tag{12}$$

where the radius $\rho = min(\frac{\sqrt{\omega_x^2+\omega_y^2}}{2}, \frac{\pi}{2})$ and the wavelet is defined by

$$\hat{\psi}(\omega_x,\omega_y) = \rho^2\hat{\phi}(\rho). \tag{13}$$

With these definitions we still have a filter bank implementation of the decompositions due to the refinable property of the scaling function

$$\begin{cases} S_{2^j}f = S_{2^{j-1}}f * h_{\uparrow 2^{j-1}} \\ W_{2^j}f = S_{2^{j-1}}f * g_{\uparrow 2^{j-1}} \end{cases} \tag{14}$$

where h, g are the 2D non-separable radial filters and the corresponding transfer functions are $H(\rho) = cos^{n+1}(\rho/2)$ and $G(\rho) = sin^2(\rho/2)$. At each scale, the image is decomposed into two components: the smoothing component and the LoG-like wavelet component. By designing the reconstruction filter to satisfy the following perfect reconstruction condition

$$H^2(\rho) + G(\rho)\tilde{G}(\rho) = 1, \tag{15}$$

the reconstruction can be obtained as

$$S_{2^{j-1}}f = S_{2^j}f * h + W_{2^j}f * \tilde{g} \tag{16}$$

where the 2D non-separable radial filter $\tilde{g}$ can be computed numerically via its Fourier transform $\tilde{G}$. Also, it is easy to check that an image can be represented as [1]

$$f(x,y) = \sum_j W_{2^j}f * \chi_{2^j}(x,y) \tag{17}$$

where χ_{2^j} is the 2D reconstruction wavelet defined by $\hat{\chi}(\rho) = \tilde{G^r}(\rho)\hat{\phi}(\rho)$.

5. COMPACT WAVELET MODELS

For image compression applications, compact representation is preferred. Whereas the scale-space technique has existed for a long time, it was the orthogonal multiresolution representation proposed by S. Mallat [10] that makes the mathematical structure of the image more explicit. This is the refinement of traditional scale-space theory. The starting point is to orthogonalize the B-spline basis, and then decompose the signal approximated at a fine scale space $S^n_{2^{i+1}}$ into a coarser scale space $S^n_{2^i}$ by imposing the orthogonal condition

$$S^n_{2^{i+1}} = S^n_{2^i} \bigoplus W^n_{2^i}. \tag{18}$$

The detailed irregular information of the signal is contained in the subspace $W^n_{2^i}$. After the B-spline basis is converted into an orthogonal basis, the two-scale relation still exists which results in an efficient pyramidal algorithm. The perfect reconstruction condition different from (15) becomes

$$|H(z)|^2 + |G(z)|^2 = 1, \tag{19}$$

Moreover, additional conditions on the filters H, G are imposed to ensure the orthogonality. There are several ways to achieve a compact multiresolution by imposing on the biorthogonal condition or interpolatory condition. Then the compact supported Daubechies wavelets, the interpolatory wavelets, the bi-orthogonal wavelets can all be derived [20]. We can show that most of these wavelet models can be derived from B-splines. A detailed study can be found in [4], [18]. Based on such observations, the implementation of the wavelet transform can be fast realized using the lattice factorization technique. For example, the transfer function of the compactly Daubechies wavelets can be factored as

$$H(z) = (\frac{1+z}{2})^N P_N(z)$$

where $(\frac{1+z}{2})^N = B^N_2(z)$ is the Nth-order of discrete B-spline with width 2 which can be realized with only addition operation. The taps of the filter P_N is much shorter than $H(z)$. Other types of the compact wavelet transforms can be treated in a similar way, which are easy for hardware implementations.

6. DISCUSS ON THE PROPERTIES OF B-SPLINE SCALE-SPACE

We can list a few advantages of the B-spline scale space over the traditional Gaussian scale-space approach with respect to the following aspects.

- **Efficiency**: The major weakness of the traditional Gaussian based scale-space is the lack of efficient algorithms. On the contrary, B-spline techniques facilitate computational efficiency. The computational complexity is scale independent.

- **Parallelism**: It seems clear enough that visual perception treats images on several levels of resolution simultaneously and that this fact must be important for the study of perception [5]. In this paper, efficient parallel structure of an image is exhibited using B-spline techniques.

- **Completeness and invertibility**: The zero-crossings or the local extrema are used as meaningful description of a signal. It is clearly important, therefore, to characterize in what sense the information in an image or a signal is captured by these primal sketches uniquely. For a Gaussian based scale space, the completeness property is guaranteed by the fingerprint theorem [11]. Later a more general proof which states that fingerprint theorem holds for any symmetry kernel was proved [14]. Therefore, *the fingerprint theorem is also true in the case of B-splines for continuous scale-space representation*.

  Differential operators have also been widely used for multiscale geometric description of images, but it has not been clear such representations are invertible. As shown in the paper, using B-splines, efficient frame algorithms can be designed to express an image from its local derivatives at dyadic scales.

- **Compactness**: For compression application, we require a representation to be as compact as possible so that an image can be represented by the corresponding primitives using less storage. In the paper, the more compact dyadic scale-space representations is proposed. We can use such representation for compression applications by combining with other techniques.

- **Causality**: A multiscale feature detection method that does not introduce new features as the scale increases is said to possess the property of causality. Such continuous causality property of the Gaussian kernel is not shared by the B-spline. However in the discrete sense, M. Aissen, I. J. Schoenberg and A. Whitney [15] had given a sufficient and necessary condition on the generating function $H(z) = \sum_{n=-\infty}^{\infty} h(n)z^n$ of a discrete scale-space kernel h to guarantee the number of local extrema or zero-crossings in $f_{out} = h * f_{in}$ does not exceed the number of local extrema or zero-crossings in f_{in}. It is easy to verify that the discrete B-spline kernel satisfies such a condition [1]. Therefore, *the causality property still holds for discrete B-spline filtering in the discrete sense*.

7. SUMMARY

In this paper, we propose the B-spline kernel as an alternative to the Gaussian kernel for scale-space representations. We formulate various forms of multiscale representations using the B-spline bases. The intrinsic relation between the wavelet models and the classical scale-space approach is exhibited. In particular, it is shown that B-splines can be used for the factorization of the wavelet transforms. As a result, fast implementation of wavelet transforms can be realized. The discussions on the properties of the B-spline scale-space have shown that the B-spline kernels have many advantages over the traditional Gaussian kernel and are well suitable for multiscale visual modeling.

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