

TOEPLITZ AND HANKEL MATRIX APPROXIMATION USING STRUCTURED APPROACH

Arnab K. Shaw, Srikanth Pokala

Ramdas Kumaresan

EE Dept., Wright State Univ.
Dayton, OH 45435

EE Dept., Univ. of Rhode Island
Kingston, RI 02881

ABSTRACT

Algorithms are presented for least-squares approximation of Toeplitz and Hankel matrices from noise corrupted or ill-composed matrices, which may not have correct structural or rank properties. Utilizing Carathéodory's Theorem on complex number representation to model the Toeplitz and Hankel matrices, it is shown that these matrices possess specific row and column structures. The inherent structures of the matrices are exploited to develop a computational algorithm for estimation of the matrices that are closest, in the Frobenius norm sense, to the given noisy or rank-excessive matrices. Simulation studies bear out the effectiveness of the proposed algorithms providing significantly better results than the state-space methods.

1. INTRODUCTION

Toeplitz and Hankel matrices occur in many engineering and statistical applications [1]. In statistical signal processing, for example, the covariance matrix of a stationary random process ideally has a Toeplitz structure. However, the ideal matrix structure can be expected only when the true covariance values are known or if the infinite length noiseless data sequence is known from which asymptotic covariance values can be calculated [4]. In practice, however, one has only finite-length noisy sequences to work with, and the estimated covariance matrix, depending on the method of estimation, may or may not retain the Toeplitz structure. Even if the computed covariance matrix has a Toeplitz form it may have a higher rank than required from the underlying signals, and in that case the problem would be to remove the excess rank by finding the closest Toeplitz matrix with the correct rank. Similarly, Hankel matrices, composed of complex sinusoids, occur in certain control and signal processing applications, but are usually formed in practice from noisy data which may destroy their properties in addition to affecting the rank. The problem would then be to estimate the closest Hankel matrix by imposing certain row/column

structures on a model based on available prior information. Earlier work on these problems has primarily consisted of using Singular Value Decomposition [4, 5, 8], where only the rank information of the underlying signal is used.

We use a theorem due to Carathéodory in complex analysis [3], in addition to the rank information of the underlying signal to form a model which imposes certain row and column structures on the Toeplitz and Hankel matrices. This facilitates the development of algorithms, along the lines of [7], for minimization of the Frobenius norm between the noisy and the model matrices, and a consequent reconstruction of the matrices.

2. TOEPLITZ MATRIX APPROXIMATION

An $N \times N$ Hermitian Toeplitz matrix $\tilde{\mathbf{R}}$ is defined by only n terms, $\tilde{r}_0, \tilde{r}_1, \dots, \tilde{r}_{N-1}$, which are possibly complex (except $\tilde{\mathbf{R}}_{11} \triangleq \tilde{r}_0$), and which form the first column of $\tilde{\mathbf{R}}$. A theorem of Carathéodory [3] regarding the trigonometric problem in complex analysis states that for such a sequence of complex numbers, there exist certain *distinct* constants on the unit circle $t_i, i = 1, \dots, p$ and certain positive weights $a_i, i = 1, \dots, p$, where $a_i > 0, |t_i| = 1$, and $1 \leq p \leq N$, such that

$$\tilde{r}_k = \sum_{i=1}^p a_i t_i^k, \quad \text{for } k = 0, \dots, N-1 \quad (1)$$

Using the above theorem, the Hermitian Toeplitz matrix $\tilde{\mathbf{R}}$ can be decomposed as follows:

$$\tilde{\mathbf{R}} \triangleq \mathbf{T} \mathbf{A} \mathbf{T}^H, \quad (2)$$

where \mathbf{T} is a Vandermonde matrix of dimension $N \times p$, such that $\mathbf{T}(n, m) = t_m^{(n-1)}$ and \mathbf{A} is a $p \times p$ diagonal matrix such that $\mathbf{A}(i, i) = a_i$. But the matrix $\tilde{\mathbf{R}}$, when formed in practice from noisy data, may not have the desired Hermitian Toeplitz structure and/or have a higher rank r than required by the signal, i.e.,

$p \leq r \leq N$. The problem therefore is to estimate $\tilde{\mathbf{R}}$ by finding t_i and a_i from the following optimization criterion:

$$\min_{\{t_i\}, \{a_i\}} \|\mathbf{e}\|^2 \triangleq \min_{\{t_i\}, \{a_i\}} \|\mathbf{R} - \tilde{\mathbf{R}}\|_F^2 \quad (3)$$

where F denotes the Frobenius matrix norm.

2.1. Decoupling

Let $\mathbf{r} \triangleq \text{vec}(\mathbf{R})$ and $\tilde{\mathbf{r}} \triangleq \text{vec}(\tilde{\mathbf{R}}) = \text{vec}(\mathbf{TAT}^H) = (\mathbf{T}^* \odot \mathbf{T})\mathbf{a}$, where $\mathbf{a} = [a_1 \cdots a_p]^T$ and \odot denotes the Khatri-Rao product of matrices. Using the above in (3), the minimization problem can be restated as

$$\begin{aligned} \min_{\{t_i\}, \{a_i\}} \|\mathbf{e}\|^2 &\triangleq \min_{\{t_i\}, \{a_i\}} \|\mathbf{r} - \tilde{\mathbf{r}}\|_2^2 \\ &= \min_{\{t_i\}, \{a_i\}} \|\mathbf{r} - (\mathbf{T}^* \odot \mathbf{T})\mathbf{a}\|_2^2 \end{aligned} \quad (4)$$

Now if we assume that we know the constants $\{t_i\}$, then the least-squares estimate of \mathbf{a} is given by

$$\hat{\mathbf{a}} = (\mathbf{T}^* \odot \mathbf{T})^\# \mathbf{r}. \quad (5)$$

Plugging this back in (5), we get

$$\begin{aligned} \min_{\{t_i\}} \|\mathbf{e}\|^2 &= \min_{\{t_i\}} \|\mathbf{r} - (\mathbf{T}^* \odot \mathbf{T})(\mathbf{T}^* \odot \mathbf{T})^\# \mathbf{r}\|_2^2 \\ &= \min_{\{t_i\}} \|(\mathbf{I}_{N^2} - \mathbf{P}_{(\mathbf{T}^* \odot \mathbf{T})})\mathbf{r}\|_2^2 \end{aligned} \quad (6)$$

where $\mathbf{P}_{(\cdot)}$ denotes the projection matrix. It is evident that the above reformulated problem depends only on $\{t_i\}$. According to a result in [2], if we obtain $\{t_i\}$ by optimizing the criterion in (7) and then compute \mathbf{a} from (6), the resulting estimates would be the same as the global minima of the original criterion in (4). Therefore the problem has now been reduced to that estimating only the non-linear parameters $\{t_i\}$ and then reconstructing the Toeplitz matrix from $\mathbf{P}_{(\mathbf{T}^* \odot \mathbf{T})}\mathbf{r}$.

2.2. Reparameterization

We now reparameterize the minimization criterion in (7) by the coefficients $\{b_i\}$ of a polynomial, whose roots are assumed to be the constants $\{t_i\}$, and then show that the error-vector has a quasi-linear relation to $\{b_i\}$, which facilitates the formulation of an iterative algorithm to estimate the polynomial coefficients, and hence the complex constants. Define $B(z) = \sum_{i=0}^{p-1} b_i z^{-i}$, whose roots are assumed to be $\{t_i\}$. Using matrices formed from the coefficients of $B(z)$, it is now deemed necessary to find the basis space orthogonal to $(\mathbf{T}^* \odot \mathbf{T})$ so that the error criterion in (7) can be reparameterized in terms of b_i . Construct an $(N-p) \times N$ matrix

\mathbf{B} as

$$\mathbf{B} \triangleq \begin{bmatrix} b_p & \cdots & b_0 & 0 \\ & \ddots & & \ddots \\ 0 & & b_p & \cdots & b_0 \end{bmatrix}, \quad (8)$$

such that \mathbf{B}^H is orthogonal to \mathbf{T} , i.e., $(\mathbf{B}^H)^H \mathbf{T} = \mathbf{0}$. Similarly \mathbf{B}^T is orthogonal to \mathbf{T}^* , i.e., $(\mathbf{B}^T)^H \mathbf{T}^* = \mathbf{0}$. Since the Carathéodory theorem requires the constants $\{t_i\}$ to be distinct, both \mathbf{T} and \mathbf{B} would be full-rank matrices such that

$$\text{rank}(\mathbf{B}^H) + \text{rank}(\mathbf{T}) = N, \quad (9)$$

and it follows from a theorem on projection matrices [6] that

$$\mathbf{P}_{\mathbf{B}^H} + \mathbf{P}_{\mathbf{T}} = \mathbf{I}_N, \quad \text{and similarly} \quad (10)$$

$$\mathbf{P}_{\mathbf{B}^T} + \mathbf{P}_{\mathbf{T}^*} = \mathbf{I}_N. \quad (11)$$

The required matrices are now constructed from \mathbf{B}^H and \mathbf{B}^T as

$$\mathbf{B}^T \otimes \mathbf{B}^H \quad N^2 \times (N^2 - 2Np + p^2) \quad (12)$$

$$\mathbf{B}^T \otimes \mathbf{T} \quad N^2 \times (Np - p^2) \quad (13)$$

$$\mathbf{T}^* \otimes \mathbf{B}^H \quad N^2 \times (Np - p^2) \quad (14)$$

Note that all the above are full-rank matrices and are orthogonal to each other and also to $\mathbf{T}^* \odot \mathbf{T}$. Also, their ranks (including that of $\mathbf{T}^* \odot \mathbf{T}$) sum up to $N^2 - (p^2 - p)$, which means that an additional full-rank matrix \mathbf{D} , of size $N^2 \times (p^2 - p)$, and orthogonal to each of the above is required to form the complete subspace orthogonal to $\mathbf{T}^* \odot \mathbf{T}$. It turns out however that it is not possible to find a \mathbf{D} which depends only on \mathbf{B}^H and \mathbf{B}^T , but not on \mathbf{T} and \mathbf{T}^* . It may be noted that rank dimension of \mathbf{D} ($p^2 - p$) is relatively small compared to N^2 . Hence we ignore the \mathbf{D} and develop an algorithm using only the remaining matrices. Simulation performance indicate that the algorithm performs quite well with this approximation.

2.3. Iterative Algorithm

Using the above matrices, and again applying the projection theorem of matrices, the error-vector of (7) can be written as

$$\mathbf{e} = (\mathbf{P}_{(\mathbf{B}^T \otimes \mathbf{B}^H)} + \mathbf{P}_{(\mathbf{B}^T \otimes \mathbf{T})} + \mathbf{P}_{(\mathbf{T}^* \otimes \mathbf{B}^H)} + \mathbf{P}_{\mathbf{D}})\mathbf{r}. \quad (15)$$

As mentioned in the previous section, we ignore the $\mathbf{P}_{\mathbf{D}}$ term and instead try to minimize the squared norm of the following vector,

$$\begin{aligned} \mathbf{e}' &= (\mathbf{P}_{(\mathbf{B}^T \otimes \mathbf{B}^H)} + \mathbf{P}_{(\mathbf{B}^T \otimes \mathbf{T})} + \mathbf{P}_{(\mathbf{T}^* \otimes \mathbf{B}^H)})\mathbf{r} \\ &= ((\mathbf{I}_N - \mathbf{P}_{\mathbf{B}^T}) \otimes \mathbf{P}_{\mathbf{B}^H} + \mathbf{P}_{\mathbf{B}^T} \otimes \mathbf{I}_N)\mathbf{r} \\ &= [(\mathbf{I}_N - \mathbf{P}_{\mathbf{B}^T}) \otimes \mathbf{W}(b_i)] [\mathbf{I}_N \otimes \mathbf{B}]\mathbf{r} \\ &\quad + [\mathbf{W}(b_i^*) \otimes \mathbf{I}_N] [\mathbf{B}^* \otimes \mathbf{I}_N]\mathbf{r}, \end{aligned} \quad (16)$$

where (10) and (11) have been used in the simplification and $\mathbf{W}(b_i) \triangleq \mathbf{B}^H(\mathbf{B}\mathbf{B}^H)^{-1}$ and $\mathbf{W}(b_i^*) \triangleq \mathbf{B}^T(\mathbf{B}^*\mathbf{B}^T)^{-1}$. The $(\mathbf{I}_N \otimes \mathbf{B})\mathbf{r}$ part of the first term in the above equation could be manipulated as

$$\begin{aligned} (\mathbf{I}_N \otimes \mathbf{B})\mathbf{r} &= \begin{bmatrix} \mathbf{B} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_N \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{R}_1 \\ \vdots \\ \mathbf{R}_M \end{bmatrix} \mathbf{b} \triangleq \mathbf{R}^1 \mathbf{b} \end{aligned} \quad (18)$$

where $\mathbf{b} = [b_0 \ b_1 \ \cdots \ b_p]$, and similarly the $(\mathbf{B}^* \otimes \mathbf{I}_N)\mathbf{r}$ part of the second term of (17) could be written as

$$\begin{aligned} &\begin{bmatrix} b_p^* \mathbf{I}_N & \cdots & b_0^* \mathbf{I}_N & \mathbf{0} \\ & \ddots & & \\ \mathbf{0} & & b_p^* \mathbf{I}_N & \cdots & b_0^* \mathbf{I}_N \end{bmatrix} \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_M \end{bmatrix} \\ &= \begin{bmatrix} b_p^* \mathbf{r}_1 + \cdots + b_0^* \mathbf{r}_{p+1} \\ b_p^* \mathbf{r}_2 + \cdots + b_0^* \mathbf{r}_{p+1} \\ \vdots \\ b_p^* \mathbf{r}_{M-p} + \cdots + b_0^* \mathbf{r}_M \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{r}_{p+1} & \cdots & \mathbf{r}_1 \\ \vdots & \ddots & \vdots \\ \mathbf{r}_M & \cdots & \mathbf{r}_{M-p} \end{bmatrix} \mathbf{b} \triangleq \mathbf{R}^2 \mathbf{b} \end{aligned} \quad (19)$$

The reparameterized error-vector can thus be written as

$$\begin{aligned} \mathbf{e}' &= [((\mathbf{I}_N - \mathbf{P}_{\mathbf{B}^T}) \otimes \mathbf{W}(b_i)) \mathbf{R}^1 + (\mathbf{W}(b_i^*) \otimes \mathbf{I}_N) \mathbf{R}^2] \mathbf{b} \\ &\triangleq \mathbf{Z} \mathbf{b} \end{aligned} \quad (20)$$

which can be seen to be quasi-linear in \mathbf{b} . The minimization of the squared error-norm of \mathbf{e}' , given by $\mathbf{b}^H \mathbf{Z}^H \mathbf{Z} \mathbf{b}$, can be iteratively performed by computing the estimate of \mathbf{b} at the i th iteration as the eigenvector corresponding to the minimum eigenvalue of $\mathbf{Z}^{(i-1)H} \mathbf{Z}^{(i-1)}$, where $\mathbf{Z}^{(i-1)}$ is evaluated using the value of \mathbf{b} obtained in the previous iteration. Conjugate symmetry constraints are imposed on the polynomial $B(z)$, in an attempt to restrict the roots to fall only on the z -plane unit circle. Note that though the above algorithm has been developed only for the Hermitian Toeplitz case because of its importance, it can quite easily be extended to the general case.

3. HANKEL MATRIX APPROXIMATION

An $N \times M$ Hankel matrix $\tilde{\mathbf{H}}$ with rank r , such that $p \leq r \leq N$, has the following decomposition,

$$\tilde{\mathbf{H}} \triangleq \mathbf{T}_N \mathbf{A} \mathbf{T}_M^T, \quad (21)$$

where \mathbf{T}_N and \mathbf{T}_M are Vandermonde matrices of dimensions $N \times p$ and $M \times p$ respectively, similar to the one defined in (2), and \mathbf{A} is a $p \times p$ diagonal matrix such that $\mathbf{A}(i, i) = A_i = a_i \exp j\phi_i$. Now in order to find the parameters of the above model by minimizing the Frobenius matrix norm of the error between the given noisy \mathbf{H} and $\tilde{\mathbf{H}}$, the following optimization criterion can be used,

$$\min_{\{t_i\}} \|((\mathbf{I}_M - \mathbf{P}_{\mathbf{B}_M^H}) \otimes \mathbf{P}_{\mathbf{B}_N^H} + \mathbf{P}_{\mathbf{B}_M^H} \otimes \mathbf{I}_N) \mathbf{r}\|_2^2, \quad (22)$$

which has been derived in a similar way as in Section 2. An algorithm similar to the one for the Toeplitz case can then be utilized to obtain t_i and hence estimate $\tilde{\mathbf{H}}$.

4. SIMULATIONS RESULTS

Three simulations have been performed to test the algorithm and the results have been compared with the state-space method in [5]. For each case, the following reasonable measure, called the Per Cent Deviation (PCD), was computed at the end of each trial with independent noise realization,

$$PCD \triangleq \sqrt{\frac{\sum_{n=0}^N \sum_{m=0}^M |\tilde{r}(n, m) - \hat{r}(n, m)|^2}{\sum_{n=0}^N \sum_{m=0}^M |\tilde{r}(n, m)|^2}} \times 100 \quad (23)$$

where $\tilde{r}(n, m)$ represents the true value while $\hat{r}(n, m)$ is the estimated value. Hundred independent trials have been performed for different SNR values in each case and the average PCD values have been computed.

Rank one Toeplitz case: The 8×8 spatial correlation matrix obtained from 8 snapshots of data composed of two complex sinusoids at 0.9708 and 1.1768 with random phase noise uniformly distributed between $[-\pi, \pi]$ and additive gaussian noise has been approximated by the two methods and the results shown in Figure 1 and Table 1.

SNR dB	Proposed Average PCD	State-space [5] Average PCD
30	3.769	15.704
20	3.804	15.711
5	5.196	16.750

Rank one Hankel case: Ten samples of an exponential with an amplitude 1 and frequency $2\pi 0.1111$ were generated and complex gaussian white noise added to it. A 7×4 Hankel matrix was formed from them and both the proposed and the state-space [5] methods were applied to approximate it at different SNR's. As is evident from

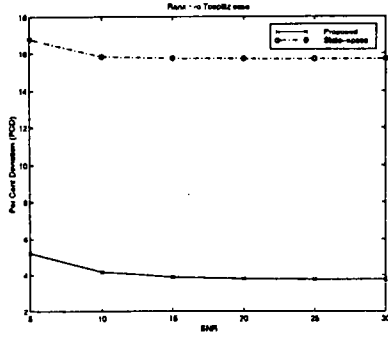


Figure 1: Rank two Toeplitz case

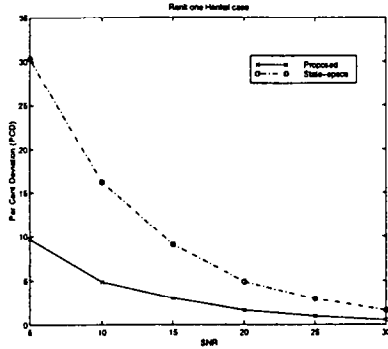


Figure 2: Rank one Hankel case

Figure 2 and Table 2, with the proposed algorithm, the estimated matrix has a significantly lower deviation from the true matrix as compared to the state-space method [5].

SNR dB	Proposed Average PCD	State-space [5] Average PCD
30	0.523	1.613
20	1.626	4.886
10	4.874	16.192

Rank two Hankel case: For this example, an 18×8 Hankel matrix was formed from gaussian-noisy data composed of two frequencies $2\pi 0.52$ and $2\pi 0.50$. Figure 3 and Table 3 show the results and demonstrate the superior performance of the structured approach of this paper.

5. REFERENCES

[1] J. P. Burg, D. G. Luenberger, and D. L. Wenger. Estimation of structured covariance matrices. *Proceedings of the IEEE*, 70(9):963–974, September 1982.

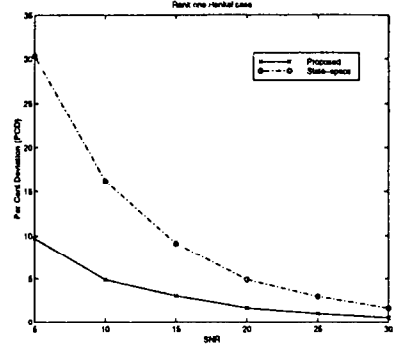


Figure 3: Rank two Hankel case

SNR dB	Proposed Average PCD	State-space [5] Average PCD
30	0.192	0.770
20	0.643	2.463
10	2.036	10.616

- [2] G. H. Golub and V. Pereyra. The differentiation of pseudoinverses and nonlinear problems whose variables separate. *SIAM Journal on Numerical Analysis*, 10(2):413–432, April 1973.
- [3] U. Grenander and G. Szego. *Toeplitz Forms and their Applications*. Chelsea Press, New York, 1986.
- [4] S. Y. Kung. A toeplitz approximation method and some applications. In *Proceedings of the International Symposium on Mathematical Theory of Networks and Systems*, August 1981.
- [5] S. Y. Kung, K. S. Arun, and D. V. Bhaskar Rao. State-space and singular-value decomposition-based approximation methods for the harmonic retrieval problem. *Journal of the Optical Society of America*, 73:1799–1811, December 1983.
- [6] C. R. Rao and S. K. Mitra. *Generalized Inverse of Matrices and its Applications*. Wiley, New York, 1971.
- [7] A. K. Shaw and R. Kumaresan. Frequency-wavenumber estimation by structured matrix approximation. *Proceedings of the 3rd. IEEE-ASSP Workshop on Spectral Estimation*, pages 81–84, November 1986.

[8] D. W. Tufts, R. Kumaresan, and Ivars Kirsteins. Data adaptive signal estimation by singular value decomposition of a data matrix. *Proceedings of the IEEE*, 70(6):684–685, June 1982.