The Generalization of the Wiener-Khinchin Theorem

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Abstract — We generalize the Wiener-Khinchin theorem. A full generalization is presented where both the autocorrelation function and power spectral density are defined in terms of a general basis set. In addition, we present a partial generalization where the density is the Fourier transform of the autocorrelation function but the autocorrelation function is defined in terms of an arbitrary basis set. Both the deterministic and random cases are considered.

1. INTRODUCTION

The Wiener-Khinchin theorem is one of the fundamental results of signal analysis. It expresses the power density in terms of the autocorrelation function. One of the reasons for its importance is that for random signals the Fourier transform may not exist but the power spectral density, obtained from the autocorrelation function does exist. In addition, the autocorrelation function is a directly measurable quantity. While one usually associates the Wiener-Khinchin theorem with random signals it holds for both random and deterministic signals.

In the introduction we briefly review the standard Wiener-Khinchin theorem and also review transforms in representations other then frequency. Subsequently we present the generalization of the Wiener-Khinchin theorem for arbitrary physical variables or representations.

Standard Wiener-Khinchin:
Deterministic Signals

We define the Fourier transform and its inverse by[2]

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int f(t)\, e^{-j\omega t}\, dt \qquad (1.1)$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int F(\omega)\, e^{j\omega t} d\omega \qquad (1.2)$$

The energy density spectrum (for frequency) defined by $|F(\omega)|^2$, can be expressed as

$$|F(\omega)|^2 = \frac{1}{\sqrt{2\pi}} \int R(\tau)\, e^{-j\omega\tau}\, d\tau \qquad (1.3)$$

where, $R(\tau)$, is the deterministic autocorrelation function,

$$R(\tau) = \frac{1}{\sqrt{2\pi}} \int f^*(t)\, f(t+\tau)\, dt \qquad (1.4)$$

This is the Wiener-Khinchin theorem for deterministic signals.

Standard Wiener-Khinchin:
Random Signals

We give a brief derivation of the standard Wiener-Khinchin theorem to motivate the method used to derive the generalization in the subsequent sections [3].

Given a function whose Fourier transform may or may not exist one defines a function for a finite interval

$$f_T(t) = \begin{cases} f(t), & -T \le t \le T; \\ 0, & \text{otherwise.} \end{cases} \qquad (1.5)$$

For random signals the power per unit frequency (power density spectrum), $S(\omega)$, is defined by

$$S(\omega) = \lim_{T\to\infty} \frac{1}{2T} E[|F_T(\omega)|^2] \qquad (1.6)$$

where $E[\ ]$ is the ensemble averaging operator. The process autocorrelation function is $R(t,\tau) = E[f^*(t)f(t+\tau)]$ and the time averaged process autocorrelation function, $R(\tau)$, is the time average of the process autocorrelation function,

$$R(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[R(t,t+\tau)]\, dt \qquad (1.7)$$

The time autocorrelation function is

$$\mathcal{R}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f^*(t)\, f(t+\tau)\, dt \qquad (1.8)$$

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[1] Work supported by the Office of Naval Research under contract No. N00014-97-1-0058. Also supported in part by the NASA JOVE and the NSA HBCU/MI programs.

[2] All integrals go from $-\infty$ to ∞ unless otherwise noted. Multiple integrals are denoted with a single integral sign, the number of integrals is noted by the number of differentials.

If the system is ergodic then $R(\tau) = \mathcal{R}(\tau)$ Now, the Fourier transform of f_T,

$$F_T(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} f(t)\, e^{-j\omega t}\, dt \tag{1.9}$$

and using Eq. (1.3) and (1.4) we have

$$|F_T(\omega)|^2 = \frac{1}{\sqrt{2\pi}} \int R_T(\tau)\, e^{-j\omega\tau}\, d\tau \tag{1.10}$$

$$R_T(\tau) = \frac{1}{\sqrt{2\pi}} \int f_T^*(t)\, f_T(t+\tau)\, dt \tag{1.11}$$

In Eq. (1.10) take the ensemble average of both sides and then the time average to obtain one form of the Wiener-Khinchin theorem.

$$S(\omega) = \frac{1}{\sqrt{2\pi}} \int R(\tau)\, e^{-j\omega\tau}\, d\tau \tag{1.12}$$

with

$$R(\tau) = \lim_{T\to\infty} \frac{1}{2T} E[R_T(\tau)] \tag{1.13}$$

If the explicit form for $R_T(\tau)$ is inserted in Eq. (1.13) we have that

$$R(\tau) = \frac{1}{\sqrt{2\pi}} E[\lim_{T\to\infty} \frac{1}{2T} \int f_T^*(t)\, f_T(t+\tau)\, dt] \tag{1.14}$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f^*(t)\, f(t+\tau)\, dt \tag{1.15}$$

In going from Eq. (1.14) to Eq. (1.15) we assumed ergodicity. Also, note that while it is not the case that $\int f_T^*(t)\, f_T(t+\tau)\, dt$ equals $\int_{-T}^{T} f^*(t)\, f(t+\tau)\, dt$, they do become equal in the limit of $T \to \infty$.

Basis Sets [1,2]

If we have physical variable a associated with the basis set (transformation matrix), $u(a,t)$, then the transform, $F(a)$, and its inverse in the a representation is[3]

$$F(a) = \int f(t)\, u^*(a,t)\, dt \tag{1.16}$$

$$f(t) = \int F(a)\, u(a,t)\, da \tag{1.17}$$

The basis sets are generally obtained as the solution to the eigenvlaue problem for an associated Hermitian operator, $\mathcal{A}$,

$$\mathcal{A}\, u(a,t) = a\, u(a,t) \tag{1.18}$$

[3] For the sake of simplicity we consider self reciprocal bases. Extension to non-reciprocal bases is straightforward. Also, integrals are assumed to go over the region defined by the range of the variables.

The transformation matrix satisfies orthogonality and completeness

$$\int u(a,t)\, u^*(a,t')\, da = \delta(t-t') \tag{1.19}$$

$$\int u(a,t)\, u^*(a',t)\, dt = \delta(a-a') \tag{1.20}$$

2.GENERALIZATION: DETERMINISTIC SIGNALS

We first give the generalization of the deterministic autocorrelation function $R(\tau)$. It can be expressed in two different forms,

$$R(\tau) = \int f^*(t)\, u(\mathcal{A},\tau)\, f(t)\, dt \tag{2.1}$$

$$= \int f^*(t)\, u^*(a,t') u(a,t) u(a,\tau)\, f(t')\, da\, dt'\, dt \tag{2.2}$$

In Eq. (2.3) $u(\mathcal{A}(t),\tau)$ means that for the variable a in $u(a,\tau)$ we substitute the operator $\mathcal{A}$. The generalization of the Wiener-Khinchin theorem for deterministic signals is then

$$|F(a)|^2 = \int R(\tau)\, u^*(a,\tau)\, d\tau \tag{2.3}$$

We now prove Eq. (2.3) with $R(\tau)$ given by Eq. (2.2). Consider [1,2]

$$\int R(\tau)\, u^*(a,\tau)\, d\tau \tag{2.4}$$

$$= \int f^*(t)\, u^*(a',t') u(a',t) u(a',\tau) u^*(a,\tau)\, f(t')\, da'\, dt'\, dt\, d\tau \tag{2.5}$$

$$= \int f^*(t)\, u^*(a',t') u(a',t) \delta(a-a')\, f(t')\, da'\, dt' \tag{2.6}$$

$$= \int f^*(t)\, u^*(a,t') u(a,t)\, f(t')\, da'\, dt' \tag{2.7}$$

$$= |F(a)|^2 \tag{2.8}$$

which proves our assertion.

It now remains to prove the equivalence of Eq. (2.1) with Eq. (2.2). Consider[4]

$$u(\mathcal{A},\tau)\, f(t) = u(\mathcal{A},\tau) \int F(a)\, u(a,t)\, da \tag{2.9}$$

$$= \int F(a)\, u(a,\tau) u(a,t)\, da \tag{2.10}$$

[4] We use the general theorem that for any function, $g(\mathcal{A})$, $g(\mathcal{A}) u(a,t) = g(a) u(a,t)$ [1,2]

$$= \int f(t')\, u^*(a,t')\, u(a,\tau) u(a,t)\, da\, dt' \qquad (2.11)$$

Hence, multiplying this by $f^*(t)$ and integrating we obtain Eq. (2.2).

3. GENERALIZATION: RANDOM SIGNALS

We define, respectively, the generalized autocorrelation function and power density for the variable a by

$$R(\tau) = \lim_{\Omega\to\infty} \eta(\Omega) \int E[\, f_\Omega^*(t)\, u(\mathcal{A},\tau)\, f_\Omega(t)\,]\, dt \qquad (3.1)$$

$$S(a) = \lim_{\Omega\to\infty} \eta(\Omega) E[|F_\Omega(a)|^2] \qquad (3.2)$$

where Ω is a region and $\eta(\Omega)$ is a function of that region chosen so that the transform of the random function exists

$$F_\Omega(a) = \int f_\Omega(t)\, u^*(a,t)\, dt = \int_\Omega f(t)\, u^*(a,t)\, dt \qquad (3.3)$$

and so that the limit in Eq. (3.1) and Eq. (3.2) exist. The generalization of the Wiener-Khinchin theorem for stochastic signals is then

$$S(a) = \int R(\tau)\, u^*(a,\tau)\, d\tau \qquad (3.4)$$

To prove Eq. (3.4) we follow the same basic idea that is used to derive the standard Wiener-Khinchin theorem. Consider

$$|F_\Omega(a)|^2 = \int R_\Omega(\tau)\, u^*(a,\tau)\, d\tau \qquad (3.5)$$

where $R_\Omega(\tau)$ is the deterministic autocorrelation function

$$R_\Omega(\tau) = \int f_\Omega^*(t)\, u(\mathcal{A},\tau)\, f_\Omega(t)\, dt \qquad (3.6)$$

Taking the ensemble average in Eq. (3.5) and the appropriate limits we have that

$$S(a) = \lim_{\Omega\to\infty} \eta(\Omega) E[|F_\Omega(a)|^2] \qquad (3.7)$$

$$= \lim_{\Omega\to\infty} \eta(\Omega) E[\int R_\Omega(\tau)\, u^*(a,\tau)\, d\tau] \qquad (3.8)$$

$$= \int \lim_{\Omega\to\infty} \eta(\Omega) E[R_\Omega(\tau)]\, u^*(a,\tau)\, d\tau \qquad (3.9)$$

$$= \int R(\tau) u^*(a,\tau)\, d\tau \qquad (3.10)$$

which is Eq. (3.4) above.

4. CHARACTERISTIC FUNCTION OPERATOR

We now give a generalization that involves the characteristic function. In this generalization the characteristic function is defined in terms of an arbitrary basis set, but the density is given as the Fourier transform of the characteristic function. The characteristic function here plays the same role as the autocorrelation function. For the Fourier basis set the characteristic function and autocorrelation function are the same. Again, we derive our results for both the deterministic and random case.

The Fourier transform of a density is the characteristic function. In our case the density is $|F(a)|^2$ and therefore the characteristic function is[5]

$$M(\theta) = \int |F(a)|^2\, e^{j\theta a}\, da \qquad (4.1)$$

with

$$|F(a)|^2 = \frac{1}{2\pi} \int M(\theta) e^{-j\theta a}\, d\theta \qquad (4.2)$$

The characteristic function can always be written as

$$M(\theta) = \int f^*(t)\, e^{j\theta\mathcal{A}} f(t)\, dt \qquad (4.3)$$

Consider the right hand side of Eq. (4.2) and use Eq. (4.3) for $M(\theta)$ to obtain

$$\frac{1}{2\pi} \int M(\theta) e^{-j\theta a}\, d\theta \qquad (4.4)$$

$$= \frac{1}{2\pi} \int f^*(t)\, e^{j\theta\mathcal{A}} f(t) e^{-j\theta a}\, dt\ d\theta \qquad (4.5)$$

$$= \frac{1}{2\pi} \int F^*(a') u^*(a',t) e^{j\theta\mathcal{A}} F(a'') u(a'',t) e^{-j\theta a}\ dt\ d\theta\, da'\, da'' \qquad (4.6)$$

$$= \frac{1}{2\pi} \int F^*(a') u^*(a',t) e^{j\theta a''} F(a'') u(a'',t) e^{-j\theta a}\ dt\ d\theta\, da'\, da'' \qquad (4.7)$$

$$= \int F^*(a') F^*(a) \delta(a-a')\, da' \ = |F(a)|^2 \qquad (4.8)$$

This result can be seen as a partial generalization when compared to that given by Eq. (2.3).

[5] For this section we use the standard 2π convention for characteristic functions.

For the random case start with

$$|F_\Omega(a)|^2 = \frac{1}{2\pi}\int M_\Omega(\theta)\, e^{-j\theta a}\, d\theta \tag{4.9}$$

where

$$M_\Omega(\theta) = \int f_\Omega^*(t)\, e^{j\theta\mathcal{A}}\, f_\Omega(t) \tag{4.10}$$

Taking the ensemble average and appropriate limit we have that

$$S(a) = \frac{1}{2\pi}\int M(\theta)\, e^{-ja\theta}\, d\theta \tag{4.11}$$

with

$$M(\theta) = \lim_{\Omega\to\infty} \frac{1}{2T}\int E[f_\Omega^*(t)\, e^{j\theta\mathcal{A}}\, f_\Omega(t)]\, dt \tag{4.12}$$

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5. EXAMPLES

Example 1: Frequency

It can be readily verified that if we take the standard Fourier basis

$$u(\omega, t) = \frac{1}{\sqrt{2\pi}}\, e^{j\omega t} \tag{5.1}$$

then all the standard Wiener-Khinchin results are obtained when inserted into our generalization.

Example 2: Scale

The scale basis, $\gamma(c,t)$, and operator, $\mathcal{C}$, are respectively

$$\gamma(c,t) = \frac{1}{\sqrt{2\pi}}\frac{e^{jc\ln t}}{\sqrt{t}} \quad , \quad t \geq 0 \tag{5.2}$$

$$\mathcal{C} = \frac{1}{2j}\left(t\frac{d}{dt} + \frac{d}{dt}t\right) \tag{5.3}$$

The scale transform, $D(c)$, and inverse are

$$D(c) = \int_0^\infty f(t)\, \gamma^*(c,t)\, dt \tag{5.4}$$

$$f(t) = \int D(c)\, \gamma(c,t)\, dc \tag{5.5}$$

Using Eq. (2.2) we find that

$$R(\tau) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f^*(t)\, f(\tau t)\, dt \tag{5.6}$$

It can be verified directly that indeed

$$|D(c)|^2 = \int_0^\infty R(\tau)\, \gamma^*(c,\tau)\, d\tau \tag{5.7}$$

These equations can be considered as generalizations of the Wiener-Khinchin theorem for deterministic signals scale. For the random case one defines

$$f_\Omega(t) = \begin{cases} f(t), & e^{-T} \leq t \leq e^T; \\ 0, & \text{otherwise.} \end{cases} \tag{5.8}$$

and the autocorrelation is

$$R(\tau) = \frac{1}{\sqrt{2\pi}} E[\lim_{T\to\infty} \frac{1}{e^T - e^{-T}}\int f_\Omega^*(t)\, f_\Omega(t\tau)\, dt] \tag{5.9}$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{T\to\infty} \frac{1}{e^T - e^{-T}}\int_{e^{-T}}^{e^T} f^*(t)\, f(t\tau)\, dt \tag{5.10}$$

For the characteristic function method, we have, using Eq. (4.3)

$$M(\theta) = \int_0^\infty f^*(t)\, e^{j\theta\mathcal{C}}\, f(t)\, dt \tag{5.11}$$

$$= \int_0^\infty f^*(e^{-\theta/2}t)\, f(e^{\theta/2}t)\, dt \tag{5.12}$$

Again, it can be directly verified that

$$|D(c)|^2 = \frac{1}{2\pi}\int M(\theta)\, e^{-j\theta c}\, d\theta \tag{5.13}$$

6. CONCLUSION

We have presented two generalizations of the Wiener-Khinchin theorem. The first generalization is a full generalization where both the autocorrelation function and power density are defined in terms of a general basis set. In the second generalization the density is the Fourier transform of the characteristic function but the characteristic function is defined in terms of an arbitrary basis set. Both generalizations reduce to the standard Wiener-Khinchin result for the Fourier basis set.

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