ESTIMATION OF FM MODULATION OF MULTI-COMPONENT SIGNALS FROM THE FOURIER PHASE

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ABSTRACT

Spectral phase is a quantity which is normally discarded in analyzing signals. In this paper, the concept of a complex time-frequency representation is presented in which the rows are narrow bandpass filters and the columns are broadband Fourier spectra. Methods are developed which exploit the spectral phase of the surface to recover the FM modulating function of an FM modulated tone and an FM modulated multi-component harmonic signal.

1.0 INTRODUCTION

In the literature, there are frequently fundamental principles which are loosely stated or which become loosely or improperly remembered. These principles then find their way back into the literature in the form of theorems and results which are much weaker than they should be. An example of this is the principle that the Fourier transform. It is often stated that the Fourier transform of a rectangularly windowed signal is the spectrum of the signal convolved with a sinc function. What is really meant is that the magnitude of the observed spectrum is somehow related to the magnitude of the signal spectrum and the magnitude of a sinc function. Since most spectral processes ignore the spectral phase, this loose formulation is generally adequate.

The purpose of this paper is to describe processes which take advantage of the spectral phase to obtain a processing gain or to obtain results which can not be obtained from the spectral magnitude alone. Applications of spectral phase were first published by the authors in several internal publications published in the mid 1980's (e.g. [3], [4], [5]). A major result in these papers was the cross-power spectral estimation (CPS) algorithm. In this algorithm, the signal is channelized by considering the short-time Fourier transform as a time-varying process. In this representation, the complex sonogram is a bank of narrow-band filters which are assumed to be sufficiently narrow to separate the components of a multi-component signal environment. In the CPS application, each signal component is assumed to be a stationary tone. The tone is detected by applying FM discrimination or instantaneous frequency transform (IFT) to each of the channels. Since the tones are assumed to be stationary, the estimates may be averaged in time to produce an improved detection procedure. The process admits an interpolation method which basically estimates the frequency of the original tones from the position of the Fourier transform bulges and the differentiated phase estimate provided by the IFT. This process works extremely well, and has proven to be superior to normal power-spectrum methods for detecting and estimating the frequencies of weak tones in noise.

The process was improved by Umesh and Nelson [6] by incorporating the Kay windowing function. With this improvement, the CPS algorithm satisfies the Cramer-Rao bound and is therefore has some claim to optimally.

In this paper, we relax the condition of stationarity and investigate the problem of spectral phase-based methods for parameterizing two signal types. The first of these is a single linear FM modulated tone in noise. The second is a frequency modulated multi-component signal whose spectrum at any time consists of the union of narrow-band harmonic components. The time varying harmonic structure assumption would seem to be quite restrictive, but, in fact, any signal which is nearly periodic may be expected to have such a structure.

2.0 TIME-VARYING FOURIER TRANSFORM

We start by defining a time-varying finite Fourier transform. We assume that we have a signal $S(t)$, We can partition time into a sequence of intervals

$$I_n = \left[\left(n-\frac{1}{2}\right)L, \left(n+\frac{1}{2}\right)L\right). \quad (1)$$

The time-varying Fourier transform is then given by

$$\hat{S}_n(\omega, W) = \int_{I_n} S(t) W(nL-t) e^{-i\omega(t-nL)} dt, \quad (2)$$

where the W is an arbitrary windowing function whose support is the interval $\left(\frac{-L}{2}, \frac{L}{2}\right]$. We can rewrite (2) by letting

$$\mathcal{W}(t) = W(t)e^{i\omega t}. \tag{3}$$

If we consider the function

$$S_{\mathcal{W}}(t) = S(t) * \mathcal{W}(t), \tag{4}$$

where * denotes convolution, we see that

$$\hat{S}_n(\omega, W) = S_{\mathcal{W}}(nL). \tag{5}$$

This means that the time-varying Fourier transform (2) may be interpreted as the decimated output of an FIR filter applied to the original signal. We will refer to the time-varying Fourier transform at frequency ω as the channel at frequency ω. With this interpretation of the Fourier transform, the phase may be predicted and used to recover signal parameters. If the Fourier transform in equation (2) is discrete and finite, the resulting time-frequency (TF) surface is sampled on a discrete lattice. If the windowing function is the rectangular window, this lattice is maximal in the sense that, for each such lattice, the values on that lattice are a minimal spanning set from which the signal may be reconstructed exactly and uniquely by concatenating the inverse Fourier transforms over each time lattice point.

3.0 RECOVERY OF A POLYNOMIAL FM MODULATED TONE.

We now consider a simple polynomial FM modulated tone and give two simple algorithms for recovery of the FM modulation. A polynomial FM modulated signal may be represented as

$$F(t) = A\exp\left(i\left(\varphi + \left(\sum_{n=1}^{N} \omega_n t^n\right)\right)\right). \tag{6}$$

The phase of F may be differentiated by the normal delay-conjugate-multiply (DCM) discrimination process of multiplying the signal by the complex conjugate of the delayed signal to produce the recursion

$$F^{(n)}(t, \varepsilon) = F^{(n-1)}(t)F^{(n-1)*}(t-\varepsilon),\ n = 1, 2, \ldots, \tag{7}$$

where $F^{(0)}(t, \varepsilon) = F(t)$. The phase derivatives may then be computed as

$$\Phi^{(n)}(t) = \lim_{\varepsilon \to 0} \frac{\arg(F^{(n)}(t, \varepsilon))}{\varepsilon^n}, \tag{8}$$

where $\Phi(t) = \arg(F(t))$, and $\Phi^{(n)}$ is the n^{th} time derivative.

For sampled signals, the limit in (8) can not be computed, but the ω_n may be computed if the degree of the phase polynomial in (6) is known. The ω_n may be computed recursively by first solving for

$$\omega_N = \frac{1}{\varepsilon^N N!}\arg(F^{(N)}(t, \varepsilon)). \tag{9}$$

Since $\arg(F^{(n)}(t, \varepsilon))$ is a polynomial involving only ω_k, for $k = n, \ldots, N$, ω_n may be recovered by substituting the known values of ε, t, and ω_k, $k = n+1, \ldots, N$ into the expression for $F^{(n)}(t, \varepsilon)$. In the absence of noise, the degree N of the phase polynomial may be estimated since the expected value of ω_N calculated by equation (9) is independent of t, and therefore constant.

For linear FM modulated tones ($\omega_n = 0$, $n > 2$), the phase polynomial may be recovered by fixing ε and Fourier transforming the resulting time signal $F^{(1)}(t, \varepsilon)$. The resulting spectrum, $\hat{F}^{(1)}(\omega)$ has an expected bulge at $\omega = 2\varepsilon\omega_2$.

4.0 RECOVERY OF AN FM MODULATED HARMONIC SIGNAL.

The assumption in the above discussion is that the signal is a modulated single tone. Under this assumption, the signal has only one component at any time, and the instantaneous frequency represents the signal's frequency. If the signal has multiple components, the full-band FM discrimination process will result in cross terms since all signal components mix when the nonlinear process is applied to the signal. In this case, the methods of the previous section may not be used to recover the FM modulation.

It is well known that any periodic signal has a harmonic spectrum in which the spectral energy is concentrated in the narrow-band harmonics

$$\Omega_n = n\Omega_1, \tag{10}$$

where Ω_n is the n^{th} harmonic, and Ω_1 is the fundamental repetition frequency of the periodic signal. If the periodicity of the signal changes with time, under certain circumstances, the harmonic relationship

$$\Omega_n(t) = n\Omega_1(t) \tag{11}$$

is satisfied. An example of such a situation is a pulse frequency modulated (PFM) impulse train

$$F(t) = \sum \delta(t_0 + nP(t)), \tag{12}$$

where $P(t)$ is the slowly time-varying pulse period.

For signals satisfying (11), we see that the harmonics are phase locked. Indeed, if the fundamental has frequency $\Omega_1(t)$ at time t and frequency $\Omega_1(t_0) + \Delta\Omega$ at time $t + \Delta t$ then the n^{th} harmonic has frequencies $n\Omega_1(t)$ and $n(\Omega_1(t) + \Delta\Omega)$ at time t and $t + \Delta t$ respectively. If we normalize by dividing by the frequency of the n^{th} component, we have in the limit

$$\frac{\Omega_n'(t)}{\Omega_n(t)} = \psi(t), \tag{13}$$

a function of t alone. We may assume that between the harmonics, the expected spectral energy is zero, so the spectral phase may be chosen arbitrarily, resulting in the general relationship is

$$\frac{\Omega'(t)}{\Omega(t)} = \psi(t). \tag{14}$$

We now derive a method for estimating the FM modulation of a multi-component harmonic signal from the phase of the complex TF representation. For simplicity, we present here only the case of a linear FM modulated harmonic structure. The result generalizes to harmonic structures with general phase functions $\Phi(t)$. The assumption that the signal components are linearly chirped is equivalent that the phase of each component may be represented as a polynomial of degree two in time with constant coefficients. In general signals are not constant linear chirps for all time. In this case, we assume that the signal may be represented locally as linear FM chirps. In this case, we may assume that the phase coefficients are slowly varying. By (14), we can factor an ω_n out of the phase function to represent the signal as

$$F(t) = \sum A_n \exp(i(\varphi_n + \omega_n(t + \kappa t^2))), \tag{15}$$

where A_n, φ_n, ω_n, and κ are assumed to be very slowly changing functions of time.

We wish to solve (15) for κ, and the function $F(t)$ is the observable. By (5). we see that, applying the time-varying Fourier transform to (15), results in the condition that each channel of the Fourier transform contains at most one signal component at any given time, as long as κ is not too large and the transform has sufficient resolution to resolve the signal components. This resolution requirement is a joint time-frequency constraint which is subject to a condition similar to the Heisenberg uncertainty principle.

We now apply a DCM process similar to (7) to each channel of the time-varying Fourier transform, to get

$$\mathcal{F}_\nu(\omega, W) = \hat{F}_\nu(\omega, W)\hat{F}^*{}_{\nu-\varepsilon}(\omega, W), \tag{16}$$

For each ω, $\hat{F}_\nu(\omega, W)$ is assumed to be dependent on at most one signal component, so the quadratic surface resulting from the DCM process contains no cross-terms. Furthermore, each component of $\mathcal{F}_\nu(\omega, W)$ has the local form

$$\mathcal{F}_\nu(\omega, W) = A_\nu(\omega)A_{\nu-\varepsilon}(\omega)e^{i\omega(\varepsilon + \kappa(2\varepsilon\nu - \varepsilon^2))}, \tag{17}$$

where ν is the time variable, and $A_\nu(\omega)$ is an amplitude term, which is assumed to be slowly time-varying. Note that (17) is a local expansion where the signal representation at each point on the time lattice is assumed to be evaluated at time $\nu = 0$. This representation simplifies the representation.

We wish to solve for κ, noting that the exponent of (17) is essentially the derivative of the signal phase with respect to time. There are several possible methods, but, since κ is assumed to be a function of time, the desired solution should involve as little integration in the time dimension as possible We present four methods.

Method 1:

We first Fourier transform with respect to ω to get

$$\hat{\mathcal{F}}_\nu(\zeta, W) = \int_\omega A e^{i\omega(\varepsilon + \kappa(2\varepsilon\nu - \varepsilon^2))} e^{-i\omega\zeta} d\omega, \tag{18}$$

where $A = A_\nu(\omega)A_{\nu-\varepsilon}(\omega)$.

$\hat{\mathcal{F}}_\nu(\zeta, W)$ has expected bulge at $\varepsilon + \kappa(2\varepsilon\nu - \varepsilon^2)$. Since ε and ν are known, κ may be recovered.

Method 2:

Normalize the argument by dividing the exponent of $\mathcal{F}_\nu(\omega, W)$ by ω. the surface can then be averaged over frequency and solved for κ.

$$\kappa = \frac{1}{(2\varepsilon\nu - \varepsilon^2)} \arg\left\{\int_\omega A e^{i(\varepsilon + \kappa(2\varepsilon\nu - \varepsilon^2))} d\omega\right\} - \varepsilon, \tag{19}$$

where $A = A_\nu(\omega)A_{\nu-\varepsilon}(\omega)$. In this solution, the delay ε must be small. Since the representation is local, the representation (19) must be evaluated at $\nu = 0$ on each of the time lattice points.

Method 3:

If we differentiate with respect to ω and average with respect to ω, κ may be recovered as

$$\kappa = \frac{1}{\delta(2\varepsilon\nu - \varepsilon^2)} \arg\left\{\int_\omega A e^{i\delta(\varepsilon + \kappa(2\varepsilon\nu - \varepsilon^2))} d\omega\right\} - \delta\varepsilon, \tag{20}$$

where $A = A_\nu(\omega)A_{\nu-\varepsilon}(\omega)A_\nu(\omega-\delta)A_{\nu-\varepsilon}(\omega-\delta)$.

Method 4:

If we differentiate with respect to ν, the Fourier transform with respect to ω may be computed as

$$\int_\omega A e^{i\omega(\kappa(2\varepsilon\delta))} e^{-i\omega\zeta} d\omega, \tag{21}$$

where $A = A_\nu(\omega)A_{\nu-\varepsilon}(\omega)A_{\nu-\delta}(\omega)A_{\nu-\varepsilon-\delta}(\omega)$. The transform (21) has an expected bulge at $\kappa(2\varepsilon\delta)$, from which κ may be easily estimated.

5.0 SIMULATIONS AND CONCLUSIONS

Samples of test data were prepared using MATLAB. These data represented one long vector of data consisting of five segments. Each segment consisted of a linear FM modulated sine wave, with different chirp rates. The FM modulating functions of the data were computed using the methods outlined in section 3.0 and were found to accurately represent the modulating functions which were synthesized.

The FM modulated pulse trains were constructed from the original modulated sine waves by half-wave rectifying the function

$$\tilde{F}(t) = abs(F(t) - 0.9) + (F(t) - 0.9), \qquad (22)$$

where $F(t)$ was the original FM modulated sine wave. The various multi-component methods were applied to verify that they can recover the modulation. The results presented in the figures represent an example of the application of method 2.

It has been verified that the methods using the phase of the time-varying Fourier transform can be very effective in extracting signal information which can be extremely difficult to recover by conventional methods.

06.0 BIBLIOGRAPHY

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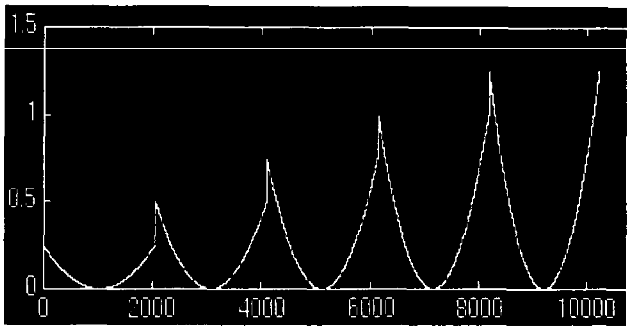

Figure 1 Time-varying phase of 5 signal segments with different chirp rates.

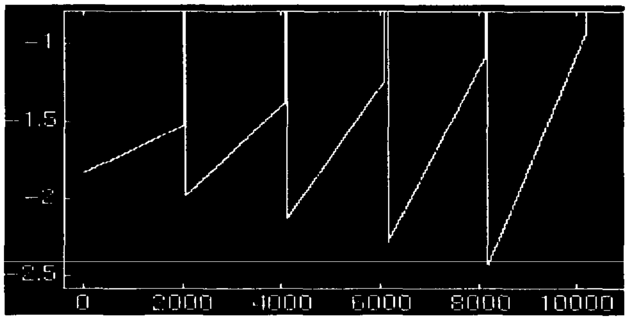

Figure 2 Time-varying frequency of 5 signal segments in figure 1.

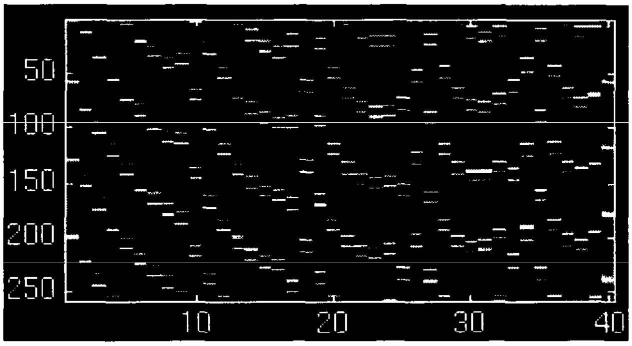

Figure 3 Time vs. time plot of 5 segments of FM modulated pulse trains with different FM rates

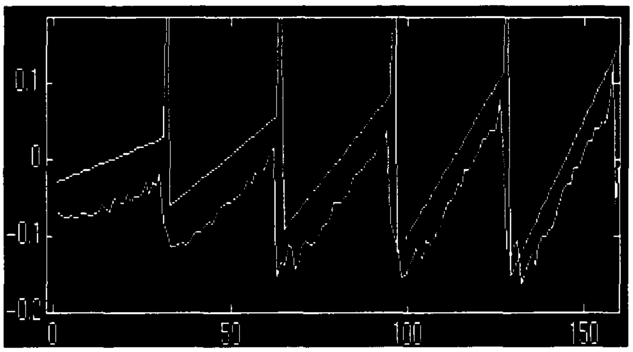

Figure 4 Original instantaneous pulse frequency (upper trace). Recovered instantaneous pulse frequency (lower trace.)