On The Relationship Between $1/f$ And α-Stable Processes

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ABSTRACT

[1] *$1/f^{\beta}$-type spectral behavior has received considerable attention in the past few years because it arises from a wide range of nature phenomena. In this paper we show that a $1/f^{\beta}$ process that can be expressed as a discretized fractional integral of white noise, follows an α-stable model if $\beta < 1$, while if $\beta > 1$ the process has stationary α-stable increments. We also provide closed form expressions for the relationship between β and α. The theoretical results are verified via real ultrasound data.*

1. Introduction

$1/f^{\beta}$-type spectral behavior abounds in nature, for example, in economic time series, in biological time series, in nature landscapes, in network traffic etc. There are inherent difficulties in developing appropriate models for processes that exhibit such spectral characteristics. As the long-range dependence persists in $1/f^{\beta}$ processes, the widely used ARMA model is not suitable in this case. Some of the models developed in the past are the "superposition of Lorenzian spectra" model [9], "infinite continuous transmission line" model [10], and "fractional brownian motion" model [11]. Wavelet-based models have also been developed to analyze and synthesize approximate $1/f^{\beta}$ behavior [7],[8] under a new frequency-domain $1/f^{\beta}$ process definition. All of the above models have provided insight into $1/f^{\beta}$ spectral behavior, there are still, however, difficulties involved when they are used in the synthesis and analysis of $1/f^{\beta}$ data.

For $\beta > 1$, the spectrum is nonintegrable around origin (often called the "infrared catastrophe"), which reflects inherent nonstationarity of the underlying process. For $\beta \leq 1$, although the spectrum is nonintegrable in its tail, (often referred to as the "ultraviolet catastrophe"). the underlying process is stationary. Both, "fractional brownian motion" and "wavelet-based" models, by being based on the assumption that the underlying process has finite variance, can only approximate a true $1/f^{\beta}$ spectral behavior.

The α-stable distribution ($0 < \alpha < 2$) is a heavy tailed distribution, and has been well known for its lack of second- and higher-order moment. The Gaussian distribution is a well known member of this family ($\alpha = 2$). Many phenomena can be well described under the α-stable distribution framework [2], [3], [4], [5]. There have been several empirical results in the literature linking $1/f^{\beta}$ spectral to α-stable processes. The increments of cardiac beat-to-beat intervals have been shown to exhibit an $1/f^{\beta}$ spectral behavior, and furthermore, their histogram is well described by a stable distribution [16]. The increments of infrared scenes data, which exhibit a $1/f$ behavior, were also demonstrated to follow well an α-stable model [15], as opposed to the Gaussian model assumed by the traditionally used fractional Brownian motion model.

In this paper we establish that a class of $1/f^{\beta}$ process that can be expressed as a discretized fractional integral of white noise follow an $\alpha-$stable distribution if $\beta \leq 1$, while, if $\beta > 1$, its increments are $\alpha-$stable distributed. We also provide the relationship between β and α. In [6] it was reported that the fractional integral of white noise is $\alpha-$stable. It is, however, the discretization of the fractional integral model that gives rise to the α-stable result. Our theoretical developments are verified via ultrasound rf-echoes that appear to follow the $1/f^{\beta}$ model that we consider here.

2 Mathematical Background

2.1 α-Stable Distributions

Stable distributions are the only class of distributions that can be the limit distributions of sums of i.i.d random variables (Generalized central limit theorem). Their density function does not have a closed form, thus

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they are usually described by their characteristic function [2]:

$$\Phi(\omega) = E\{exp(j\omega x)\} \quad (1)$$

$$= \begin{cases} exp(ja\omega - \gamma|\omega|^{\alpha}(1 - j\eta \tan\frac{\pi\alpha}{2} sign(\omega))) & \alpha \neq 1 \\ exp(ja\omega - \gamma|\omega|^{\alpha}(1 - j\eta\frac{2}{\pi} ln|\omega| sign(\omega))) & \alpha = 1, \end{cases}$$

where $\alpha \in (0,2]$ is the characteristic exponent, $\eta \in [-1,1]$ is the symmetry index, $\gamma > 0$ is the spread parameter, and a is the location parameter. A symmetric α-stable $S\alpha S$ process is described by the characteristic function given in (1), where $\eta = 0$.

α-stable processes ($0 < \alpha < 2$) have only finite moments of order p, where $-1 < p < \alpha$, while their second- or higher-order moments are not defined. Fractional-order moments, however, exist and their definitions can be found in [2]. Similarly, for such process the covariance is not defined. Its fractional-order equivalent, the covariation is defined (only for stable processes with $\alpha > 1$) and its definition can be found in [2].

2.2 $1/f^{\beta}$-type spectrum

The underlying process of an $1/f^{\beta}$-type spectrum, i.e., $x(t)$, can be formulated in the scenario of fractional calculus [13]. In that sense, white noise, $w(t)$, is the β-order fractional derivative of $x(t)$, i.e.,:

$$\frac{d^{\beta}}{dt^{\beta}} x(t) = w(t), \quad (3)$$

where β is a real number, normally between 0 and 2. Based on (3), $x(t)$ can be obtained via the Riemann-Liouville fractional integral [13]:

$$x(t) = \frac{1}{\Gamma(\beta/2)} \int_0^t (t - t')^{\beta/2-1} w(t') dt', \quad (4)$$

where $\Gamma(\cdot)$ is the Gamma function. From (4), $x(t)$ can be viewed as the output of a linear time-invariant system excited by white noise, whose impulse response is:

$$G(t) = \frac{t^{\beta/2-1}}{\Gamma(\beta/2)} u(t), \quad (5)$$

where $u(t)$ is the unit step function. The Laplace transform of $G(t)$ equals $1/s^{\beta/2}$, thus the power spectrum of $x(t)$ is $1/f^{\beta/2}$. It should be noted that for $\beta < 2$, $G(0)$ is not finite, which results in infinite variance in the underlying process. This will cause a problem when this linear filtering model is used in synthesizing $1/f^{\beta}$ data. The fractional brownian motion model bypasses this problem [11] by having finite variance. As a result, however, it can only approximate $1/f^{\beta}$ behavior. Since we are not going to use (4) in synthesizing $1/f^{\beta}$ processes, but rather in deriving the statistics of such a process, the model of (4) will serve our purpose.

3 Relationship Between $1/f^{\beta}$ Spectrum and α-Stable Distribution

I) $\beta < 1$

(2) Let $x(t)$ be an $1/f^{\beta}$ process with $\beta < 1$. Discretizing the integral in (4) yields:

$$x(t) = \sum_{j=1}^{N} w_j G(t - t_j) \quad (6)$$

where $t_j \in [-T, T]$ (uniform i.i.d), w_j (i.i.d) are, respectively, the location and the magnitude of the j-th pulse, and t_j and w_j are independent with each other. Then the characteristic function of $x(t)$ is

$$\begin{aligned} \Phi_x(\rho) &= E\{e^{j\rho x(t)}\} = \prod_{j=1}^{N} E\{e^{j\rho w_j G(t-t_j)}\} \\ &= [E\{e^{j\rho w_j G(t-t_j)}\}]^N \\ &= \left[\int_{-T}^{T}\int_{-\infty}^{\infty} e^{j\rho w G(t-t')} f(w) \frac{1}{2T} dw dt'\right]^N \\ &= \left[1 + \frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{\infty} \left[e^{j\rho w G(t-t')} - 1\right] f(w) dw dt'\right]^N . \end{aligned} \quad (7)$$

Let $N \to \infty$, $T \to \infty$, while keeping $m = N/2T$ constant. Taking into account the limit formula $\lim_{T\to\infty}(1 + \frac{s}{T}) = e^s$, (7), yields :

$$\Phi_x(\rho) = e^{Z(\rho)}, \quad (8)$$

where

$$Z(\rho) = m \int_{-\infty}^{\infty} f(w) dw \int_{-\infty}^{\infty} \left[e^{j\rho w G(t-t')} - 1\right] dt'. \quad (9)$$

Substituting in (8) $G(t - t')$ with

$$G(t - t') = \begin{cases} 0, & t' \geq t \\ \frac{(t-t')^{\lambda-1}}{\Gamma(\lambda)}, & t' < t, \end{cases} \quad (10)$$

where $\lambda = \beta/2$, yields:

$$\begin{aligned} Z(\rho) &= m \int_{-\infty}^{\infty} f(w) dw \int_{-\infty}^{t} \left[e^{j\rho w \frac{(t-t')^{\lambda-1}}{\Gamma(\lambda)}} - 1\right] dt' \\ &= m \int_{0}^{\infty} f(-w) dw \int_{-\infty}^{t} \left[e^{j\rho w \frac{(t-t')^{\lambda-1}}{\Gamma(\lambda)}} - 1\right] dt' \\ &+ m \int_{0}^{\infty} f(w) dw \int_{-\infty}^{t} \left[e^{j\rho w \frac{(t-t')^{\lambda-1}}{\Gamma(\lambda)}} - 1\right] dt' \end{aligned} \quad (11)$$

Changing the variable of intergation to $s = \frac{w}{\Gamma(\lambda)}(t - t')^{\lambda-1}$ in (11), yields :

$$
\begin{aligned}
& Z(\rho) \\
= & -m \int_0^\infty f(-w)dw \int_0^\infty \left[e^{-j\rho s} - 1\right] \alpha s^{-\alpha-1}\Gamma(\lambda)^{-\alpha} w^\alpha ds \\
- & \; m \int_0^\infty f(w)dw \int_0^\infty \left[e^{j\rho s} - 1\right] \alpha s^{-\alpha-1}\Gamma(\lambda)^{-\alpha} w^\alpha ds \\
= & \frac{-m\alpha}{\Gamma(\lambda)^\alpha} \int_0^\infty [f(-w) + f(w)] w^\alpha dw \int_0^\infty (\cos \rho s - 1) s^{-\alpha-1} ds \\
+ & \; j\frac{m\alpha}{\Gamma(\lambda)^\alpha} \int_0^\infty [f(-w) - f(w)] w^\alpha dw \int_0^\infty (\sin \rho s) s^{-\alpha-1} ds
\end{aligned}
\tag{12}
$$

where

$$
\alpha = \frac{1}{1-\lambda} = \frac{2}{2-\beta}. \tag{13}
$$

Evaluating the integrals in (12) yields [14]:

$$
\begin{aligned}
Z(\rho) &= c \cdot q \cdot \Gamma(-\alpha) \cdot \cos\left(\frac{\alpha\pi}{2}\right) \cdot \rho^\alpha \\
&\quad + jc \cdot r \cdot \Gamma(-\alpha) \cdot \sin\left(\frac{\pi\alpha}{2}\right) |\rho|^\alpha sign(\rho) \\
&= c \cdot q \cdot \Gamma(-\alpha) \cdot \cos(\frac{\alpha\pi}{2}) \cdot \rho^\alpha \cdot \\
&\quad (1 + j\frac{r}{q} \cdot \tan(\frac{\alpha\pi}{2}) \cdot sign(\rho)) \quad (for\ \alpha \neq 1),
\end{aligned}
\tag{14}
$$

where

$$
c = \frac{m\alpha}{\Gamma(\lambda)^\alpha}. \tag{15}
$$

$$
q = \int_0^\infty [f(-w) + f(w)] w^\alpha dw, \tag{16}
$$

$$
r = \int_0^\infty [f(w) - f(-w)] w^\alpha dw.
$$

By comparing (14) with (2) one can see that $x(t)$ follows an α-stable distribution, with dispersion equal to $c \cdot q \cdot \Gamma(-\alpha) \cdot \cos(\frac{\alpha\pi}{2}) \cdot \rho^\alpha$. symmetry index $-r/q$. and location parameter 0.

II) $1 < \beta < 3$

If $1 < \beta < 3$ the underlying process $x(t)$ is nonstationary. This can be seen by viewing this process as the integration of the stationary $1/f^{\beta'}$ process, $x'(t)$, with $\beta' = \beta - 2 < 1$ [7], i.e.,

$$
x(t) = \int_0^t x'(t). \tag{17}
$$

As t increases, the dispersion of $x(t)$ diverges. Since the spectral exponent of $x'(t)$ equals $\beta - 2$, the increment process $x'(t)$ falls under case (I), thus is α-stable.

4 Simulation results

Our theoretical results are justified by studying ultrasound breast data. The ultrasound rf echo follows the model [12]:

$$
x(t) = w(t) * h(t) + n(t), \tag{18}
$$

where $w(t)$ is a random process that models the tissue response; $h(t)$ is a deterministic kernel that models the ultrasonic system response (transducer and attenuation); and $n(t)$ is noise or modeling error. The ultrasonic system in the frequency domain is a bell shaped function which peaks at the transducer center frequency.

The data used in this section are breast images, and were obtained using a flat linear array transducer with a nominal center frequency of 7.5 MHz, on a clinical imaging system UltraMark-9 HDI Advanced Technology Laboratories. The sampling rate of the data was 20 MHz. Based on line segments of 250 samples, taken along the axial direction, the power spectrum was estimated. In a loglog plot of the power spectrum of various segments, it can be seen that, in the low-frequency range, successive segments of the ultrasound echo exhibit a $1/f^\beta$ trend, with β changing between segments (see Fig. 1). Thus, at low frequencies, where the spectral contribution of transducer is almost flat, the rf echo spectrum exhibits a clear $1/f^\beta$ behavior. Based on (18), this means that $w(t)$ is $1/f^\beta$. The tissue response is widely modeled as a Poisson process. Thus, the tissue response echo is consistent with the model of (6). According to the main result of this paper, depending on β, $w(t)$, or its increments, should be α-stable distributed. If the noise term in (18) can be ignored, the filtered $w(t)$, i.e., $x(t)$, or its increments, will also be α-stable. The parameter α will be the same as in $w(t)$, since linear filtering does not change α. From the same segments over which the $1/f^\beta$ trend was validated, the α-stable model was tested using the method of [15]. Fig. 2 shows the loglog plot of the characteristic function versus ω (see (2)), whose slope provides an estimate for α.

The almost perfect lines of Fig. 2 confirm that the corresponding data follows the α-stable model. The expected α would be $\alpha = \frac{2}{2-\beta}$ if $x(t)$ is $1/f^\beta$ with $\beta < 1$, or $\alpha = \frac{2}{4-\beta}$ for the increment of $x(t)$ if $x(t)$ is $1/f^\beta$

with $\beta > 1$. In the cases included in Fig. 2, there is good agreement between the expected α parameter and estimated one. The differences, which can be sometimes observed between estimated and expected α, can be attributed to the noise term $n(t)$, or errors in the estimation of α due to the short data records used.

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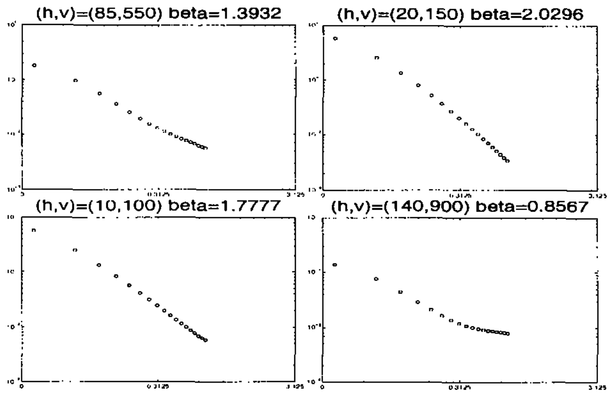

Figure 1. Low frequency part of ultrasound power spectrum, estimated from various windows of lenth 256 samples along the axial direction.

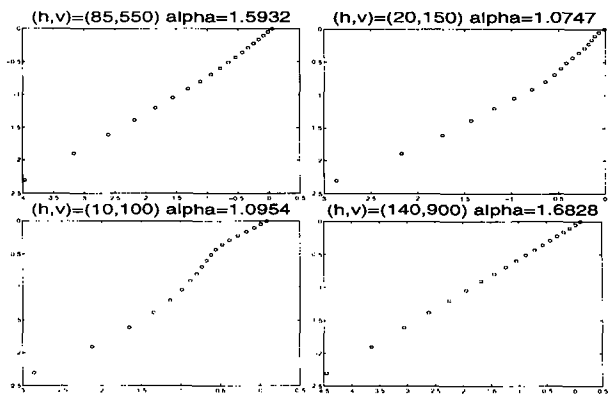

Figure 2. Estimation of α from the same windows used in Fig. 1.