A GEOMETRICAL FRAMEWORK FOR THE DETERMINATION OF AMBIGUOUS DIRECTIONS IN SUBSPACE METHODS

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ABSTRACT

In signal subspace parameter estimation techniques, like MUSIC, degradations may occur due to parasite peaks in the spectrum, which may be connected to high sidelobes in the beam pattern or to ambiguities themselves. The aim of this paper is to study the presence of ambiguities in an array of given planar geometry. We propose a general framework for the analysis and thus we obtain a generalisation of results given in recent publications [2], [3] for rank one and two ambiguities. For rank $k \geq 3$ ambiguities the study is restricted to linear arrays, for which we derive original and synthetic results. We present a geometrical construction able to determine all the ambiguous directions which can appear for a given linear array. The method allows determination of any rank ambiguities and for each ambiguous direction set, the rank of ambiguity is obtained. The search is exhaustive. Application of the method requires no assumption for the linear array and is easy to implement. An example is detailed for a non uniform linear array.

1. INTRODUCTION

Resolution is not the unique criterion in the performance evaluation of sources localisation techniques. Degradations may occur due to parasite peaks in the spectrum, which may be connected to high sidelobes in the beam pattern (sometimes referred as quasi-ambiguities) or to ambiguities themselves. These ambiguities arise when the array manifold intersects itself or when a manifold vector can be written as a linear combination of two or more manifold vectors [1]. The aim of this paper is to study the presence of ambiguities in the use of subspace method.

We propose a general framework for the analysis and thus we obtain a generalisation of results given in recent publications [2], [3] for rank one and two ambiguities. For rank $k \geq 3$ ambiguities the study is restricted to linear arrays, for which we derive original and synthetic results. We present a geometrical construction able to determine all the ambiguous directions which can appear for a given linear array. This is a geometrical approach closely connected to [3]. The method allows determination of any rank ambiguities and for each ambiguous direction set the rank of ambiguity is determined. The search is exhaustive. Application of the method requires no assumption for the linear array and is easy to implement.

In section 2 notations and definitions of ambiguity are introduced. In section 3 a study of rank one ambiguous arrays is presented. Section 4 depicts the main results obtained for rank two ambiguous arrays. This study is conducted for planar arrays of arbitrary geometry. In section 5, the study is restricted to linear arrays for any rank k ambiguities. The proposed method is presented.

In section 6, an example is detailed for a non uniform linear array. Section 7 includes some conclusions.

2. PROBLEM FORMULATION AND DEFINITIONS

Consider an array with M sensors receiving N narrowband signals impinging on the array from N different locations $\theta_1,\ldots,\theta_N$. Note $A(\theta_1,\ldots,\theta_N)=[a(\theta_1),\ldots,a(\theta_N)]$, the matrix which columns are the sources steering vectors, also called the array manifold vectors.

The simultaneous localisation of N sources is only possible if the array manifold vectors $a(\theta_1),\ldots,a(\theta_N)$ are linearly independent. An array is said rank k ambiguous for a set of $k+1$ directions of arrival $\theta_1,\ldots,\theta_{k+1}$ if matrix A is singular but rank k [1]. This can be written:

$$\exists \alpha_1 \neq 0,\ldots,\alpha_{k+1} \neq 0 \text{ so that } \alpha_1 a(\theta_1)+\ldots+\alpha_{k+1}a(\theta_{k+1})=0$$
$$(\alpha_1,\ldots,\alpha_{k+1}) \in C^{k+1} \quad (1)$$

3. RANK ONE AMBIGUITIES (FOR GENERAL ARRAYS)

This case occurs when one array manifold vector $a(\theta_1)$ can be written as a complex scalar multiple of another manifold vector $a(\theta_2)$ where $\theta_1 \neq \theta_2$.

$$\exists(\alpha_1 \neq 0, \alpha_2 \neq 0) \in C^2, \text{ so that } \alpha_1 a(\theta_1)+\alpha_2 a(\theta_2)=0 \quad (2)$$

In such case, the array cannot make the difference between two waves with bearings θ_1 or θ_2.

The wavefronts are supposed straight-line and on the same plane as the sensors. $\vec{k}_1$ and $\vec{k}_2$ being the ambiguous wave vectors for the array under consideration, the phase delay of signal n from sensor m to sensor one is :

$$\varphi_{mn} = \vec{k}_n . \vec{r}_m \quad (3)$$

where $\vec{r}_m$ denotes the position of the m^{th} sensor in half wavelength. Equation (2) is then equivalent to the condition :

$$\alpha_1 e^{-j\varphi_{m1}} + \alpha_2 e^{-j\varphi_{m2}} = 0 \Leftrightarrow \varphi_{m1} = \varphi_{m2} + 2n_m\pi \quad (4)$$

for $m = 1, \ldots, M$

The ambiguity condition can be written :

$$\exists p_m,\ \text{integer}\ (\vec{k}_1 - \vec{k}_2)\vec{r}_m = 2p_m\pi \quad (5)$$

with $|\vec{k}| = 2\pi/\lambda$ where λ stands for the wavelength. It can be given the following geometrical interpretation, see Fig. 1 :

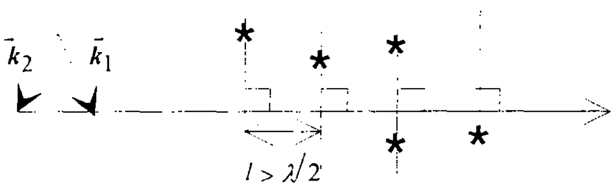

Figure 1. Stars represent some possible sensor positions for a rank one ambiguous array. The horizontal axis is defined by vectors $\vec{k}_1$ and $\vec{k}_2$.

The consequence is that, for arrays of arbitrary geometry, rank 1 ambiguities can arise if all of its sensors are located on a set of parallel lines separated by a distance $l > \lambda/2$. In the case of a linear array this result refunds the classical Shannon condition. In the general case, it establishes conditions for ambiguity and then can give the ambiguous directions [5], [7].

4. RANK TWO AMBIGUITIES (FOR GENERAL PLANAR ARRAYS)

This situation occurs when the array manifold line intersects a plane in more than two points. In such case, one manifold vector can be written as a linear combination of two others manifold vectors, which may be written:

$$\exists(\alpha_1, \alpha_2, \alpha_3) \in C^3 \quad \alpha_1 a(\theta_1) + \alpha_2 a(\theta_2) + \alpha_3 a(\theta_3) = 0$$
$$(\alpha_1 = 1) \quad (6)$$

with $a(\theta_n) = [\ldots \quad e^{-j\varphi_{mn}} \quad \ldots]^T$ and $\varphi_{mn} = \vec{k}_n . \vec{r}_m$. Sensor 1 is taken as a reference $\vec{r}_1 = \vec{0}$.

Therefore for sensor 1, $\varphi_{11} = \varphi_{12} = \varphi_{13} = 0$. The ambiguity condition (6) can thus be written:

$$1 + \alpha_2 + \alpha_3 = 0 \quad (7)$$

This relation can be interpreted geometrically in the complex plan as a triangle which sides are the steering vectors

For sensor m ambiguity condition (6) becomes:

$$e^{j\varphi_{m1}} + \alpha_2 e^{j\varphi_{m2}} + \alpha_3 e^{j\varphi_{m3}} = 0 \quad (8)$$

In the complex plan the product by $e^{j\varphi}$ is a rotation. Thus the sides $\vec{1}, \vec{\alpha}_2, \vec{\alpha}_3$ turn respectively from angles $\varphi_{m1}, \varphi_{m2}, \varphi_{m3}$ and must reconstitute a triangle according to relation (8). The length of the sides of the triangle must be the same, therefore the triangles are deducted one from another by an isometry. This isometry can be a rotation or a rotation associated to a symmetry. Thus the triangles corresponding to the different values of m belong to two sub-families, the rotation family and the rotation associated to a symmetry family. See [6] and [7] for more details on these isometry family.

The following results can then be derived [5], [7] :

1) Each rank two ambiguous array may be split in two subarrays $a^1(\theta)$ and $a^2(\theta)$, where $a^1(\theta)$ and $a^2(\theta)$ are rank one ambiguous, for three directions θ_1, θ_2 and θ_3 i.e.: $a^i(\theta_1) = a^i(\theta_2) = a^i(\theta_3)$.

2) As a consequence, the sensors for each subarray are located at the nodes of a two dimensional lattice.

3) Lattices corresponding to the two subarrays are related by an arbitrary translation. This is a simpler demonstration and a generalisation of a previous result of Lo and Marple [2].

5. RANK K AMBIGUITIES FOR LINEAR ARRAYS

By generalisation of the previous results, we infer that the sensor array can be splitten in k subarrays. In each subarray sensors are on a grid of spacing denoted a. The

k grids are translated one from another. For the first grid :

$$\vec{r}_m = a N_m \vec{v} \qquad (9)$$

where v is the unitary vector of the linear array.

Let us denote $\vec{k} = (2\pi/\lambda)\vec{u}$. If a is the largest common denominator of the inter sensor distances in a subarray, the ambiguity condition can be written [5], [7]:

$$\vec{v}(\vec{u}_i - \vec{u}_j) = n_{ij}(\lambda/a) \qquad (10)$$

Thus all the sets of vectors $\vec{u}_1,\ldots,\vec{u}_{k+1}$ which can be projected on the grid of step λ/a are ambiguous. By arbitrary translation of this grid, an infinity of ambiguous direction sets can be obtained.

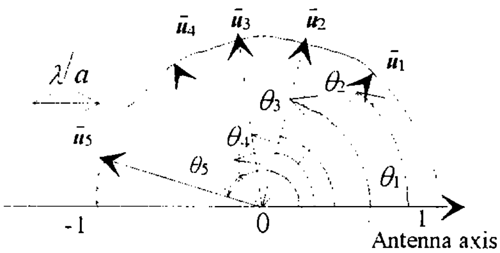

Figure 2. Determination of the ambiguous directions of arrival for a linear array.

It appears clearly on Fig. 2 that the condition for no rank k ambiguities is : $k(\lambda/a) > 2$.

This nice geometrical property is closely connected to the notion of generator set of ambiguities introduced by Proukakis and Manikas [3]. We define the generator set of ambiguity as the set $\{\vec{u}_1,\vec{u}_2,\ldots,\vec{u}_k,\vec{u}_{k+1}\}$, where $\vec{u}_1 = \vec{v}$.

In order to save space, we will now represent the generator set as in Fig. 3.

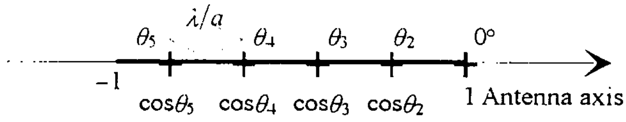

Figure 3. Representation of an ambiguous generator set.

6. GEOMETRICAL DETERMINATION OF AMBIGUOUS GENERATOR SETS

Based on the above considerations, we propose a method for the determination of the ambiguous generator sets and the corresponding rank of ambiguity for a linear array. The principle is very similar to [3]. Let us consider a linear array of M sensors.

Proposed method :

1- Compute all the inter sensor distances. Note $r_{ij} = |\vec{r}_j - \vec{r}_i|$ in half wave-length.

2- All the intersensors distances $a = r_{ij}$ smaller than 1 cannot provide ambiguities because $(\lambda/a) > 2$.

3- Consider each inter sensor distance r_{ij}, verifying condition 2- $r_{ij} > 1$, compute the corresponding generator direction set with $a = r_{ij}$ (see figure 3). The result is $\{0°,\theta_2,\ldots,\theta_l\}$, where l is the number of considered intersensor distances.

4- Split the array into subarrays so that in subarrays sensors are located on grids of step a translated one from another. The construction must be done in order to get a minimum of subarrays. Note q the number of subarrays.

5- If $q \geq l$, there is no ambiguous generator set for this value of a.

If $q < l$, then the array presents a rank q ambiguity, the ambiguous generator set is given by $\{0°,\theta_2,\ldots,\theta_l\}$.

6- Continue with the step 3- until all the intersensor distances have been taken into account.

The method is very easy to implement and requires no assumption. Thus all ambiguous direction sets are determined for the considered linear array.

This geometrical approach of the search of ambiguities in linear arrays allows us to begin a new study of sparse linear arrays. In many papers, [4], non uniformly spaced linear arrays are studied, in particular minimum redundancy and nonredundant arrays. The compromise between array span and sampling gain are discussed. We propose now to study ambiguities and quasiambiguities for these arrays. Application of the proposed method brings some enlighting results.

7. APPLICATION OF THE PROPOSED METHOD TO PROUKAKIS AND MANIKAS EXAMPLE[3]

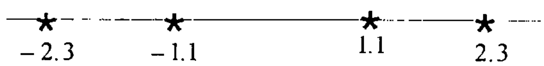

Figure 4. Sensor positions on the array in half wavelength.

In their example, three sources are located in : 0°, 55.582° and 82.505°. The considered array is a sparse linear array.

Two parasite peaks appear in the spectrum of MUSIC located in 107.719° and 137.657°. Because the array is ambiguous, the MUSIC algorithm provided five directions rather than three.

This phenomenon was not clearly explained in [3]. Application of the proposed method allows us to predict these ambiguous directions of arrival.

1- Inter sensor distances in half wavelength : $\{1.2,\ 2.2,\ 3.4,\ 4.6\}$

3- $\boxed{a = r_{12} = 1.2}$, . The possibly ambiguous directions set is given by the following construction :

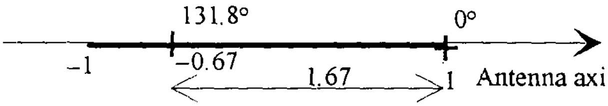

Figure 5. Determination of the possible ambiguous directions set : $\{0°, 131.8°\}$, $l = 2$.

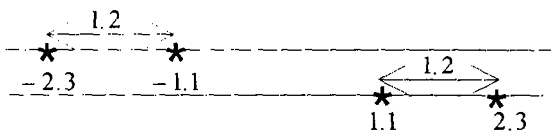

Figure 6. Splitting into subarrays, $q = 2$.

The construction gives two subarrays, $q = 2$. Because there are only two directions concerned, it can be a rank two ambiguity $(q \geq l)$. A rank two ambiguity can arise only for three or more concerned directions. We conclude that for $a = r_{12} = 1.2$, there is no ambiguity.

$\boxed{a = r_{23} = 2.2}$, the same situation gives the same conclusion.

$\boxed{a = r_{13} = 3.4}$, $\lambda/a = 2/3.4 = 0.58$.

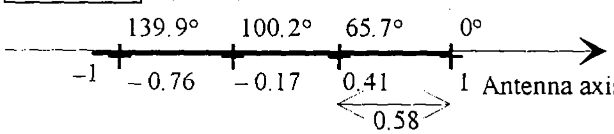

Figure 7. Determination of the possible ambiguous directions set : $\{0°, 65.7°, 100.2°, 139.9°\}$, $l = 4$.

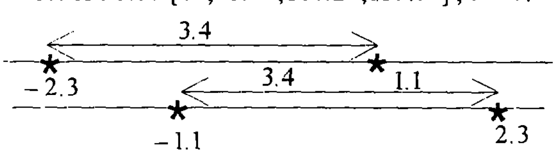

Figure 8. Splitting into subarrays, $q = 2$.

The construction demonstrates the existence of a rank two ambiguity for the considered array. The ambiguous directions set is given on Fig. 7 We verify with MUSIC that if two sources are located in any two directions previously determined, two parasite peaks appear in the two foreseen directions.

$\boxed{a = r_{14} = 4.6}$, $\lambda/a = 2/4.6 = 0.43$. The construction of the possibly ambiguous directions set, as presented on figure 4 provides $\{0°, 55.6°, 82.5°, 107.7°, 137.6°\}$. Let us construct the subarrays.

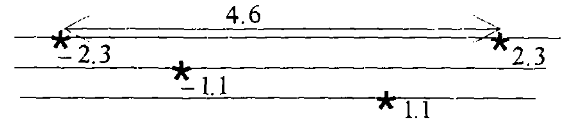

Figure 9. Splitting into subarrays, $q = 3$.

This is a rank three ambiguity. The directions of arrivals are exactly those detected in the MUSIC spectrum. It proves that the predicted ambiguous direction set is really an ambiguous set. Furthermore, we are able to say that the ambiguity is a rank three ambiguity.

8. CONCLUSION

We propose a general geometrical framework to study ambiguities for arbitrary arrays. For linear arrays, a geometrical construction is presented and is able to predict all the ambiguous directions for the considered array. The presented method opens a new way to study and design non uniformly spaced linear arrays.

9. REFERENCES

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