TIME-VARYING SPECTRUM ESTIMATORS FOR CONTINUOUS-TIME SIGNALS

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ABSTRACT

Some quadratic time-frequency representations (TFRs) may be called *time-varying spectrum estimators.* They are derived from first principles, and they turn out to be time-varying *multiwindow* spectrum estimators. In special cases they are time-varying spectrograms that may be written as Fourier transforms of lag-windowed, time-varying correlation sequences or as spectrally smoothed time-varying periodograms. These are not ad-hoc variations on stationary ideas to accommodate time variation. Rather, they are the only variations one can obtain for time-varying spectrum analysis.

1. INTRODUCTION

In this paper we derive from first principles the most general quadratic, modulation-invariant, delay-invariant, non-negative estimator of a power spectrum for continuous-time signals. We call it a *time-varying spectrum estimator,* and not surprisingly it belongs to the Cohen class of quadratic time-frequency representations (TFRs) [4]. The most general time-varying spectrum estimator is a time-varying multiwindow spectrum estimator reminiscent of the stationary discrete-time Thomson multiwindow spectrum estimator [8],[10]. Only in special cases is it truly a continuous-time-varying version of the Blackman-Tukey-Grenander-Rosenblatt spectrogram [1],[6]. This special case is especially interesting because it is obtained by factoring the kernel of the time-varying spectrum estimator into a product of diagonal and Toeplitz kernels, of which one determines time-frequency resolution and the other determines spectral smoothing for variance control.

This paper is a companion to [10], which treats time-varying spectrum estimators for discrete-time signals.

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2. QUADRATIC ESTIMATORS OF A TIME-VARYING SPECTRUM

Let $s(t)$, $t \in \mathbb{R}$, denote a complex-valued, continuous-time signal and call $P(t, \omega)$, $(t, \omega) \in \mathbb{R} \times (-\infty, \infty]$, a quadratic estimator of a time-varying spectrum, a term to be clarified in due course. In order to qualify as a quadratic estimator of a time-varying spectrum, we shall insist that $P(t, \omega)$ satisfy these three properties for all signals $s \in L_2$:

P1: Modulation Invariance. If $s(t)$ is complex frequency modulated as $s(t)e^{j\omega_0 t}$, then $P(t, \omega)$ is replaced by $P(t, \omega - \omega_0)$. This property preserves our understanding of frequency ω.

P2: Delay Invariance. If $s(t)$ is time-delayed as $s(t - t_0)$, then $P(t, \omega)$ is replaced by $P(t - t_0, \omega)$. This property preserves our understanding of time t.

P3: Quadratic and Non-Negative. The dependence of $P(t, \omega)$ on $s(t)$ is a quadratic form in a non-negative definite Hermitian kernel $Q(u, v; t, \omega)$:

$$P(t, \omega) = \int\int s^*(u) Q(u, v; t, \omega) s(v)\, du\, dv \geq 0 \quad (1)$$

$$Q(u, v; t, \omega) = Q^*(u, v; t, \omega) \;\; \forall\, (t, \omega). \quad (2)$$

This property preserves our understanding of power (or energy) and ensures that $P(t, \omega)$ scales as $|\beta|^2$ when $s(t)$ is complex scaled as $\beta s(t)$.

Let us now follow the methodology of [8] and [9] and explore the implications of properties P1–P3. Begin with P3 and enforce P1:

$$P(t, \omega - \omega_0) = \int\int s^*(u) e^{-j\omega_0 u} Q(u, v; t, \omega) e^{j\omega_0 v} s(v) du dv. \quad (3)$$

Set $\omega = 0$ and replace ω_0 by $-\omega$ to find

$$P(t, \omega) = \sum_u \sum_v s^*(u) e^{j\omega u} Q(u, v; t, 0) e^{-j\omega v} s(v)\, du\, dv. \quad (4)$$

This holds for all s iff

$$Q(u, v; t, \omega) \equiv e^{j\omega u} Q(u, v; t, 0) e^{-j\omega v}. \quad (5)$$

Now enforce $P2$:

$$P(t - t_0, \omega) = \quad (6)$$

$$\int\int s^*(u - t_0) e^{j\omega u} Q(u, v; t, 0) e^{-j\omega v} s(v - t_0)\, du\, dv.$$

Set $t = 0$ and replace t_0 by $-t$ to find

$$P(t,\omega) = \int\int s^*(u+t)e^{j\omega u}Q(u,v)e^{-j\omega v}s(v+t), \quad (7)$$

where we have made the substitution $Q(u,v) \triangleq Q(u,v;0,0)$. Even though the modulation and delay operators do not commute, this representation is invariant to the order in which they are applied to the signal.

With no constraints on $Q(u,v)$, equation (7) is a representation of the Cohen class of quadratic *time-frequency representations* (TFRs) [2],[7]. However, a more illuminating form for our purposes is

$$P(t,\omega) = \int e^{-j\omega\tau}r(\tau,t)\,d\tau = S(\omega,t), \quad (8)$$

where $r(\tau,t)$ is a deterministic *time-varying correlation sequence* and $S(\omega,t)$ is its spectrum:

$$r(\tau,t) = \int s(v+t)Q(v-\tau,v)s^*(v+t-\tau)\,dv \quad (9)$$

$$S(\omega,t) = \int r(\tau,t)e^{-j\omega\tau}\,d\tau. \quad (10)$$

This representation justifies our terminology *time-varying spectrum estimator* for $P(t,\omega)$. As the kernel $Q(u,v)$ is Hermitian, the time-varying correlation $r(\tau,t)$ is a (non-negative definite) Hermitian correlation sequence for all t, and $P(t,\omega)$ is a real and non-negative spectrum. For more general kernels, $r(\tau,t)$ is indefinite and $P(t,\omega)$ is an indefinite TFR.

In summary, the quadratic TFRs that can be called time-varying spectrum estimators have representations (7) and (8), with the kernel $Q(u,v)$ (non-negative definite) Hermitian. This is not a new result, but its derivation from first principles may be. We now turn to a more thorough investigation of time-varying spectrum estimators.

3. THE TRACE CLASS OF TIME-VARYING MULTIWINDOW SPECTRUM ESTIMATORS

If the Hermitian kernel $Q(m,n)$ is trace-class, then it may be factored as

$$Q(u,v) = \sum_i q_i^*(u)q_i(v) \quad (11)$$

with normalization

$$\int Q(v,v)\,dv = 1 = \sum_i \int |q_i(v)|^2\,dv = \sum_i \lambda_i. \quad (12)$$

Then the representation for the class of time-varying spectrum estimators is the following *time-varying multiple-window estimator:*

$$\begin{aligned} P(t,\omega) &= \sum_i \left| \int q_i(v)s(v+t)e^{-j\omega v}\,dv \right|^2 \\ &= \sum_i S_i(\omega,t), \end{aligned} \quad (13)$$

where $S_i(\omega,t)$ is the i^{th} time-varying and windowed spectrum:

$$S_i(\omega,t) = \left| \int q_i(v)s(v+t)e^{-j\omega v}\,dv \right|^2. \quad (14)$$

We may also write $P(t,\omega)$ as the Fourier transform of $r(\tau,t)$ as in equation (8), where $r(\tau,t)$ is now a multiwindow correlation sequence:

$$r(k,t) = \sum_i \int q_i(v)s(v+t)q_i^*(v-\tau)s^*(v+t-\tau). \quad (15)$$

These representations for the class of time-varying spectrum estimators are the most general that one can obtain without admitting unbounded operators. So, we say that the most general time-varying spectrum estimator is a time-varying multiple-window spectrum estimator. The great virtue of this multiwindow formulation is that $P(t,\omega)$ may be computed in real time on a finite lattice of the Nyquist band $(-\pi,\pi]$ as

$$P(t,\omega_k) = \sum_i \left| \int h_i(t-v)s(v)e^{-j\omega_k v}\,dv \right|^2, \quad (16)$$

where the filter $h_i(t)$ is a time reversal of the window $q_i(t)$:

$$h_i(t) = q_i(-t). \quad (17)$$

That is, the signal $s(t)$ may be demodulated by $\exp(-j\omega_k t)$, passed through a linear time-invariant filter $h_i(t)$, and its output squared to obtain the i^{th} component of the time-varying spectrum at time t and frequency ω_k. For this computation to be real-time, we require $h_i(t)$ to be rational.

4. TIME-VARYING SPECTROGRAMS

Now let us suppose that the Hermitian kernel $Q(u,v)$ has the factorization (diagonal, Toeplitz, diagonal):

$$Q(u,v) = w_1^*(u)w_0(v-u)w_1(v) \quad (18)$$

$$w_0(v-u) = w_0^*(u-v). \quad (19)$$

Then the time-varying spectrum estimator $P(t,\omega)$ may be written as

$$\begin{aligned} P(t,\omega) &= \int e^{-j\omega\tau}w_0(\tau)r_1(\tau,t)\,d\tau \\ &= \int_{-\infty}^{\infty} \hat{W}_0(\omega-\nu)S_1(\nu,t)\,\frac{d\nu}{2\pi}, \end{aligned} \quad (20)$$

where $r_1(\tau, t)$ is a single-window correlation sequence and $S_1(\omega, t)$ is its spectrum:

$$r_1(\tau, t) = \int w_1(v)s(v+t)w_1^*(v-\tau)s^*(v+t-\tau)\,dv \quad (21)$$

$$S_1(\omega, t) = \int r_1(\tau, t)e^{-j\omega\tau}\,d\tau \geq 0 \;\forall\, t. \qquad (22)$$

That is, $P(t,\omega)$ is the Fourier transform of a lag-windowed time-varying correlation, making it the continuous time varying version of the stationary Blackman-Tukey-Grenander-Rosenblatt spectrogram. It is also a spectrally smoothed time-varying spectrum. The beauty of equation (18) is that it allows us to design the window $w_1(v)$ for time-frequency resolution and the window $w_0(v)$ for spectral smoothing for variance control. This feature has not been evident heretofore in time-frequency spectrum analysis. Once windows are designed, then a finite-rank matrix $Q(u,v) = w_1^*(u) \cdot w_0(v-u)w_1(v)$ may be approximated as

$$Q(u,v) = \sum_i q_i^*(u)q_i(v) \qquad (23)$$

with the $q_i(u)$ solutions to the Fredholm equation

$$\int Q(u,v)q_i^*(v)\,dv = \lambda_i q_i(u). \qquad (24)$$

Then the spectrum may be computed in real time as the multiwindow spectrum estimator of equation (16). An obvious choice for $w_1(v)$ is a Gaussian pulse, thus validating Gabor's insight, and the obvious choice for $w_0(v)$ is the unit pulse response of a lowpass filter, thus validating Blackman and Tukey's insights. Another way to say it is this: we would like to design the windows $q_i(v)$ for time-frequency resolution *and* variance control. By recasting this design problem in terms of $w_0(v)$ and $w_1(v)$, we indirectly design the windows $q_i(v)$. The net effect of the windows $q_i(v)$ is the combined effects of $w_0(v)$ and $w_1(v)$.

When $w_1(v) = 1 \;\forall\, v$, then $P(t,\omega)$ is independent of t and equal to the Blackman-Tukey-Grenander-Rosenblatt spectrogram:

$$\begin{aligned} P(\omega) &= \int e^{-j\omega\tau}w_0(\tau)r(\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} \hat{W}_0(\omega-\nu)S(\nu)\,\frac{d\nu}{2\pi}, \end{aligned} \qquad (25)$$

where (for finite energy signals) $r(\tau)$ and $S(\omega)$ are stationary and finite:

$$r(\tau) = \int s(v)s^*(v-\tau)\,dv \qquad (26)$$

$$S(\omega) = \int r(\tau)e^{-j\omega\tau}\,d\tau. \qquad (27)$$

This justifies our claim that only this class of time-varying spectrum estimators should be called *time-varying spectrograms*.

5. OTHER PROPERTIES OF $P(T,\omega)$

It is natural to ask when the time-varying spectrum $P(t,\omega)$ is a nominal spectrum $S(\omega)$ carried along the time-frequency plane as

$$P(t,\omega) = S\big(\omega - \beta(t)\big). \qquad (28)$$

In this case we would call $\beta(t)$ the *carrier frequency* of the time-varying spectrum without becoming embroiled in a debate about what constitutes the *instantaneous frequency* of the signal $s(t)$. It is not hard to show that the most general signal for which equation (28) holds is the linear FM signal

$$s(t) = \exp\left[j(\phi_0 + \phi_1 t + \phi_2 t^2)\right], \qquad (29)$$

in which case the carrier frequency $\beta(t)$ is linear:

$$\beta(t) = \phi_1 + 2\phi_2 t. \qquad (30)$$

There is no other signal whose time-varying spectrum is carried as in equation (28), and there is no more general carrier frequency for a time-varying spectrum than the linear frequency $\phi_1 + 2\phi_2 t$. Except for degenerate kernels that provide no time resolution, the time-frequency spectrum $P(t,\omega)$ can never be invariant to time scaling. However, for small timescale changes of the form $s(rt)$, the linear FM signal has a time-frequency spectrum that is *approximately*

$$P(t,\omega) \cong S\Big(\frac{\omega}{\gamma} - \phi_1 - 2\phi_2\gamma t\Big). \qquad (31)$$

This is what we mean by inverse time-frequency spreading for a time-varying spectrum estimator.

6. CONCLUSIONS

Quadratic time-varying spectrum estimators are time-varying, multiwindow estimators. They are reminiscent of the stationary multiwindow estimators discovered by Thomson for discrete time. There is a subclass of time-varying spectrum estimators called time-varying spectrograms. They are reminiscent of the stationary discrete-time spectrograms discovered by Blackman, Tukey, Grenander, and Rosenblatt. In a time-varying spectrogram, time-frequency resolution is determined by one window and frequency smoothing for variance control by another.

We do not think that one should quibble with properties P1 through P3 as the minimum properties one should expect from a time-varying spectrum estimator. The resulting function of time and frequency is the Fourier transform of a time-varying multiwindow correlation or, equivalently, a time-varying multiwindow spectrum estimator. In some cases, it is a spectrally smoothed time-varying spectrum. This makes the term *time-varying spectrum estimator* seem legitimate, even though one can never define with precision what kind of time-frequency spectrum is *being estimated.* In fact, it is this difficulty that has motivated us to define instead what we mean by a time-varying spectrum *estimator* and to see what the minimum set of constraints on such an estimator produces. It produces *time-varying* versions of multiwindowed, lag-windowed, and spectrally smoothed periodograms—not as ad-hoc variations but as the most general variations one can find. This means that future work on time-varying spectrum analysis should be directed toward the design of windows which trade off time-frequency resolution and variance control to meet the objectives of the designer.

The existing theory of TFRs emphasizes the role of the so-called time and frequency marginal property. TFRs are usually designed so that the time and frequency marginals match the magnitude squared signal and its magnitude squared Fourier transform, respectively [3]. The question of marginals lies outside the line of argumentation for time-varying spectrum estimators. Of course, no time-varying spectrum estimator *can or does* match marginals. This is a fundamental property. Therefore, we have trouble with the suggestion, often made, that time-varying spectrum estimators are defective for not matching marginals. Any (non-Hermitian) quadratic TFR which *does* match marginals is matching them with something other than energy, and the TFR is something other than a time-varying spectrum estimator. In our opinion, the question of marginals is not central to the problem of time-varying spectrum analysis, and the matching of energy marginals with something other than energy is an idea of questionable value [5]. This is not to say that non-Hermitian TFRs are of no value in signal analysis where something other than energy is computed as a function of time and frequency. A catalogue of many non-Hermitian TFRs, together with their interesting properties, is contained in [11].

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