IDENTIFICATION OF ARBITRARILY TIME-VARIANT SYSTEMS

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ABSTRACT

We introduce a technique for identification of systems with arbitrarily time-variant responses from samples of their input and output signals, and without using any prior information about the dynamics of the unknown system response. Our technique is based on the use of optimized averaging filters for the estimation of time-variant second order moments. We demonstrate the utility of our approach and the quality of the resulting estimates via a numerical example.

1. INTRODUCTION

Time-variant systems are often encountered in engineering applications, ranging from communications via atmospheric and underwater fading channels, through marine seismography and array processing, to study of biological signals. The failure of conventional (i.e., stationary-based) system identification techniques to cope with rapid time variation has motivated the development of several novel approaches to identification of linear, arbitrarily time-variant systems.

In this article we introduce a technique for identification of systems with arbitrarily time-variant responses from measurements of their (stationary) input and (nonstationary) output signals, and ***without any prior assumptions about the unknown system response.*** This is in contrast to currently available procedures, all of which rely on prior statistical characterization of the time-variant system response.

The problem that sets the framework for our research is the identification of time-varying systems from input-output data, as depicted in Fig. 1, where $W(t)$ denotes the time-variant impulse response of the system we wish to identify. Here $u(t)$ is the input signal to the system, $d(t)$ is the (noise-corrupted) output, and t denotes discrete time. We adopt the so-called ***output error approach*** in which one attempts to determine the unknown $W(t)$ by setting up a 'duplicate' system with the same input $u(t)$ and with

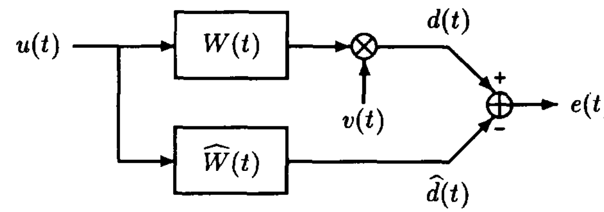


Figure 1: Problem description

impulse response $\widehat{W}(t)$, and adjusting $\widehat{W}(t)$ so that the output error

$$e(t) = d(t) - \widehat{d}(t) \tag{1}$$

is as small as possible. The synthesized output $\widehat{d}(t)$ is

$$\widehat{d}(t) = \sum_{k=0}^{M} \widehat{w}_k(t)\, u(t-k) = \widehat{W}(t)\, U(t) \tag{2a}$$

where

$$\widehat{W}(t) = [\,\widehat{w}_0(t) \;\; \widehat{w}_1(t) \;\; \ldots \;\; \widehat{w}_M(t)] \tag{2b}$$

and

$$U(t) = [\,u(t) \;\; u(t-1) \;\; \ldots \;\; u(t-M)]^T . \tag{2c}$$

Similarly,

$$d(t) = W(t)\, U(t) + v(t) \tag{3a}$$

where

$$W(t) = [\,w_0(t) \;\; w_1(t) \;\; \ldots \;\; w_M(t)] \tag{3b}$$

and where the additive (stationary) noise $v(t)$ is uncorrelated with the input signal $u(t)$.

Several solutions were suggested for this time-variant identification problem: (i) the moment estimation approach – requires average autocorrelations of certain time-variant moments [1]; (ii) the weighted least squares approach – requires selection of a weighting sequence [2]; (iii) the function series expansion approach – requires selection of basis functions [3]; and (*iv*) the state space (Kalman filter) approach – requires specification of state space model parameters [4,5].

In this paper we present a method for identification of arbitrarily time-variant linear systems whose input (i.e., the signal $u(t)$ in Fig. 1) is stationary. Thus, our technique is useful in the broad range of applications where the input is either stationary by nature (e.g., fading communication channels), or can be selected at will (to be stationary). Our method combines the moment estimation approach of [1] with a technique for efficient estimation of the required prior information directly from samples of the signals $d(t)$ and $u(t)$.

The starting point for the moment estimation approach, as discussed in [1,6], is the observation that the true response $W(t)$ satisfies the linear (so-called "Wiener-Hopf") equation

$$R_{dU}(t) = E\{d(t)\,U^*(t)\} = W(t)\,R_{UU}(t) \qquad (4)$$

where $R_{UU}(t) = E\{U(t)\,U^*(t)\}$. Moreover, when $U(t)$ is stationary the time-invariant covariance matrix $R_{UU} \equiv R_{UU}(t)$ can be estimated by conventional techniques (such as exponential averaging [7]), so that the only serious remaining challenge is the estimation of the time-variant moment $R_{dU}(t)$. The moment estimation approach of [1] constructs an optimized linear estimate of $R_{dU}(t)$ by using certain prior information about the joint statistics of $d(t)$ and $u(t)$.

We show in Sec. 2 that such prior information can be expressed in terms of two average autocorrelations: the autocorrelation $c_v(m) = E\{v(t+m)v^*(t)\}$ of the additive stationary noise $v(t)$, and the deterministic autocorrelation of the response vector $W(t)$, viz.,

$$C_W(m) = \langle\, W^*(t)\,W(t+m)\,\rangle \qquad (5)$$

where $\langle\;\rangle$ denotes time averaging. Subsequently, we show in Sec. 3 how these average autocorrelations can be efficiently estimated from samples of the signals $d(t)$ and $u(t)$. Thus, we are able to construct an optimized estimate $\widehat{W}(t)$ of the unknown arbitrarily time-variant response $W(t)$ without using any prior information about the dynamics of either the system response $W(t)$ or the additive noise $v(t)$.

2. MOMENT ESTIMATION WITH AVERAGING FILTERS

In the moment estimation approach to system identification one replaces the probabilistic moments $R_{dU}(t)$, R_{UU} in (4) by suitable estimates $\widehat{R}_{dU}(t)$, $\widehat{R}_{UU}(t)$. Since the time-invariant probabilistic covariance R_{UU} can be efficiently estimated by conventional techniques, we may assume that, as $t \to \infty$, the estimate $\widehat{R}_{UU}(t)$ has converged to its steady-state value, namely that $\widehat{R}_{UU}(t) \approx R_{UU}$. Since we are interested here only in steady-state estimation errors, we may replace the estimate $\widehat{W}(t) = \widehat{R}_{dU}(t)\,\widehat{R}_{UU}^{-1}(t)$ by a simplified version, namely $\widehat{W}(t) = \widehat{R}_{dU}(t)\,R_{UU}^{-1}$. We have shown in [6] that the dynamics of the more refined estimate $\widehat{R}_{dU}(t)\,\widehat{R}_{UU}^{-1}(t)$ converge, as $t \to \infty$, to those of the simplified version, so that these two estimates become indistinguishable in steady-state.

As described in [1], the time-variant moment estimate $\widehat{R}_{dU}(t)$ is obtained by applying an averaging filter $H(\cdot)$ to the composite (multichannel) signal $\xi(t) = d(t)\,U^*(t)$. Consequently, the simplified estimate $\widehat{W}(t) = \widehat{R}_{dU}(t)\,R_{UU}^{-1}$ is obtained by scaling the input of the averaging filter, as shown in Fig. 2.

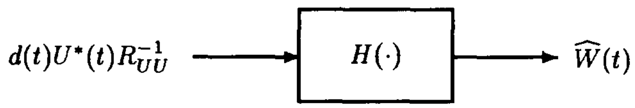


Figure 2: Obtaining $\widehat{W}(t)$ by LTI averaging.

This means that we can apply the methodology of [1] to optimize the averaging filter $H(\cdot)$ in the sense of minimizing the time- and ensemble-averaged estimation error $\mathcal{E} = \langle\, E\|\widehat{W}(t) - W(t)\|^2\,\rangle$. The explicit expressions derived in [1] for the optimized $H(\cdot)$ can now be used, provided one has prior knowledge of two average autocorrelations, viz.,

$$c_B(m) = \langle\, W(t+m)\,W^*(t)\,\rangle \qquad (6a)$$

$$c_V(m) = \langle\, E\{\Upsilon(t+m)\,\Upsilon^*(t)\}\,\rangle \qquad (6b)$$

where $\Upsilon(t) = d(t)U^*(t)R_{UU}^{-1} - W(t)$. We have shown in [6] that these average autocorrelations can be expressed in terms of $c_v(m)$, $C_W(m)$, and certain moments of the stationary input signal $u(t)$, viz.,

$$c_B(m) = \mathrm{tr}\,\{C_W(m)\} \qquad (7a)$$

$$c_V(m) = c_v(m)\,\mathrm{tr}\,\{C_U(m)\} + \mathrm{tr}\,\{\Gamma_U(m)\,C_W(m)\} \qquad (7b)$$

where

$$C_U(m) = E\{U(t)\,U^*(t+m)\} \qquad (7c)$$

$$\Gamma_U(m) = E\{U(t+m)\,U^*(t+m)\,U(t)\,U^*(t)\} - I \qquad (7d)$$

In past work, optimized averaging filters have been constructed by utilizing prior information about $C_W(m)$, $c_v(m)$ in applications where such information was readily available [1,8,9]. In contrast, we now show (in Sec. 3) how to determine this information directly from samples of $u(t)$ and $d(t)$ and nothing else.

3. ESTIMATION OF SYSTEM AND NOISE AVERAGE AUTOCORRELATIONS

The average autocorrelation of any observed signal (stationary or not) can be efficiently estimated by conventional time averaging, under suitable ergodicity assumptions (see, e.g., [10]). However, the average autocorrelations $c_v(m)$ and $C_W(m)$ cannot be directly estimated because neither $v(t)$, nor $W(t)$ are directly available. Instead, we set up a system of linear equations that relates these unknown autocorrelations to $c_d(m) = \langle E\{d(t+m)d^*(t)\}\rangle$, the average autocorrelation of the output signal $d(t)$, and to $C_\xi(m) = \langle E\{\xi^*(t)\xi(t+m)\}\rangle$, the average autocorrelation of the composite signal $\xi(t) = d(t)U^*(t)$. As established in [6], these equations are

$$\mathcal{Q}\begin{pmatrix} c_v(m) \\ \mathrm{vec}\{C_W(m)\} \end{pmatrix} = \begin{pmatrix} c_d(m) \\ \mathrm{vec}\{C_\xi(m)\} \end{pmatrix} \qquad (8)$$

where $\mathrm{vec}\{\cdot\}$ denotes the vectorization of a matrix by columns (see, e.g., [11]). The matrix $\mathcal{Q}$ consists of certain second and fourth-order moments of the input signal $u(t)$, viz.,

$$\mathcal{Q} = E\left\{\begin{pmatrix} 1 \\ \widetilde{U}(t+m)\otimes U(t) \end{pmatrix}\begin{pmatrix} 1 \\ \widetilde{U}(t+m)\otimes U(t) \end{pmatrix}^*\right\}$$

where $\sim$ denotes element by element conjugation and $\otimes$ stands for a Kronecker product (see, e.g., [11]). As mentioned earlier, the elements of $\mathcal{Q}$, as well as the average autocorrelations $c_d(m)$ and $c_\xi(m)$, can all be consistently estimated by conventional time averaging. This means that the estimation errors associated with such estimates vanish asymptotically as the length of the data record increases. Consequently, the estimates of $C_W(m)$, $c_v(m)$, and the resulting averaging filter $H(\cdot)$, can all be made as accurate as desired by increasing the length of the data record.

A particular case of special interest is when the input signal $u(t)$ is Gaussian, which allows us to collapse eq. (7),(8) into the much simpler relation

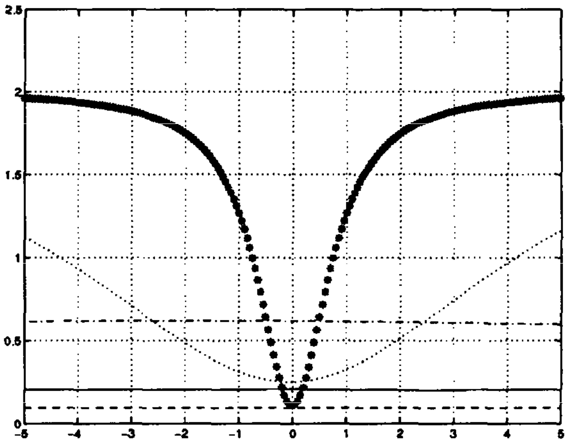
$$c_V(m) + c_B(m) = \mathrm{tr}\,\{C_\xi(m)\, R_{UU}^{-2}\} \qquad (9a)$$

$$c_V(m) = c_d(m)\ \mathrm{tr}\,\{C_U^*(m)\, R_{UU}^{-2}\} \qquad (9b)$$

Thus, we completely avoid the explicit estimation of $c_v(m)$, $C_W(m)$, and we obtain the autocorrelations $c_B(m)$, $c_V(m)$ needed to construct the optimized averaging filter $H(\cdot)$ directly from $C_\xi(m)$, $C_U(m)$ and $R_{UU} \equiv C_U(0)$. Clearly, eqs. (9) offer a significant reduction in computational cost, as compared with eqs. (7)-(8).

4. EXAMPLE

In this section we demonstrate the utility of our methodology by applying it to the same periodically-time-variant example that was considered in [1,8], namely $d(t) = w(t)\,u(t) + v(t)$, where $w(t) = \sigma_0 + 2\sigma_1\cos 2\pi f_0 t$, with $f_0 = 1/5(1+\delta)$ and $|\delta| \le \delta_{max} = 0.05$. We compare the performance of our ***prior-information-independent*** (PII) estimator with that of several estimators from [1,8], all of which exploit the periodicity of $w(t)$ and the constraint $|f_0 - 1/5| \le 1/5\,\delta_{max}$. The plots in Figs. 3,4 present the steady-state error $\mathcal{E} = \langle E|\widehat{w}(t) - w(t)|^2\rangle$ as a function of the relative frequency deviation δ: Fig. 3 is based on theoretical expressions, involving both time and ensemble averaging, while Fig. 4 presents the time-averaged error from a single sample path obtained by simulation [6]. We implement the optimized averaging filter in FIR form, and observe that the error $\mathcal{E}$ decreases (to zero, in this example) as the length L of the FIR filter increases. As is evident from the plots, our PII estimator (with $L = 100$) outperforms the other techniques. As explained in [1,8], estimators that rely on prior information can achieve *either* low estimation error *or* robustness with respect to perturbations in the priors, but never both. In contrast, our PII estimator is not subject to such a tradeoff: it always achieves the theoretical minimum for $\mathcal{E}$ as its length L increases, and this error is independent of δ.



$\mathcal{E}$

δ (%)

Figure 3: The theoretical estimation error $\mathcal{E}$ vs. the relative frequency deviation δ (in %), for: (i) finely-tuned periodic estimator $(***)$; (ii) optimized harmonic estimator $(\cdots)$; (iii) PII estimator of length $L = 10$ $(-\cdot-)$, $L = 40$ (—), and $L = 100$ $(--)$.

The empirical plots of Fig. 4 essentially coincide with the theoretical ones (Fig. 3), except for a slight increase in error of the PII estimator, which is due to finite record length effects.

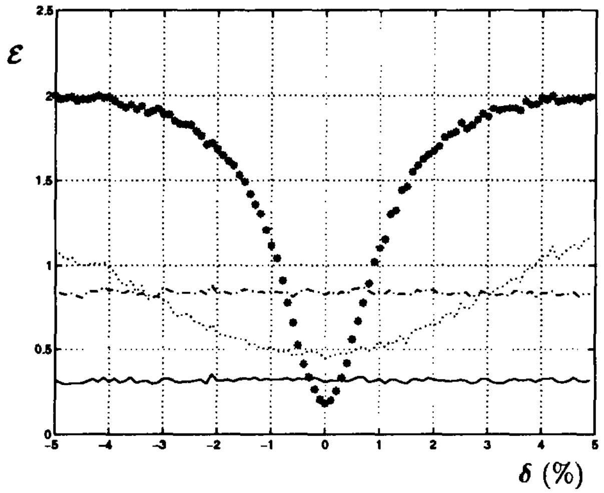


Figure 4: The empirical estimation error $\mathcal{E}$ vs. the relative frequency deviation δ (in %), for: (i) finely-tuned periodic estimator ($***$); (ii) optimized harmonic estimator ($\cdots$); (iii) PII estimator of order $L = 10$ ($- \cdot -$), and $L = 40$ (——).

5. CONCLUDING REMARKS

In this research we have demonstrated the feasibility of identifying the response of an arbitrarily time-variant linear system from samples of its input and output signals, without relying on any prior information. Our approach is based on moment estimation via optimized averaging: as explained in [1,8], optimized selection of the required averaging filters relies on availability of the two average autocorrelations $c_B(m)$ and $c_V(m)$ of eq. (6). We have shown in [6] that when the input signal is stationary such information can be determined directly from the observed input and output signals. This is in contrast to previously proposed techniques for time-variant system identification, all of which require prior information.

One important by-product of our computational procedure is an estimate of the average autocorrelation $C_W(m)$ of the unknown system response $W(t)$. This statistical characterization is essential in any method for identification of arbitrarily time-variant systems. For instance, the extended RLS algorithm of [5] relies on explicit prior knowledge of the parameters of an underlying state space model: our approach should make it possible to identify the required parameters directly from samples of the input and output signals.

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