DESIGN OF RECURSIVE FILTERS WITH CONSTANT GROUP DELAY AND CHEBYSHEV ATTENUATION

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ABSTRACT

This article presents some new results concerning recursive filters design with approximately linear phase and Chebyshev stopband attenuation. The denominator polynomial $D(z)$ of the transfer function $H(z)=N(z)/D(z)$ is used to obtain a maximally flat behavior for the delay in the passband, whereas $N(z)$ describes equiripple amplitude in the stopband. The approach under consideration is based on z-domain concepts. At the end, the paper concludes with several detailed examples and graphics showing the efficacy of the proposed technique.

1. INTRODUCTION

In many applications filters are required to satisfy certain amplitude specifications and at the same time to approximate linear-phase and/or constant group delay in the passband. A number of papers have appeared in technical literature concerning the subject of linear-phase (or approximately linear-phase) digital filters. The design of IIR transfer functions with both maximally flat and Chebyshev group delay has been studied by Thiran and others [1, 2, 8]. In all these cases, the magnitude response was monotonic and therefore not highly selective for a given order [3]. The work was extended to cover Chebyshev stopband attenuation by Unbehauen [9, 10] and by Maria and Fahmy [4]. Also, an optimization procedure has been used by Deczky, Saramaki, and Neuvo [1, 6].

It is well known that IIR filters satisfying the desired amplitude and phase specifications can be designed in two parts [7, 9] : first, the denominator is determined to satisfy the phase requirements, then the mirror-image or antimirror-image numerator is designed to achieve the magnitude requirements of the filter. The same approach is used in [7] where a new class of an all-pole transfer function for IIR synthesis is described. In this case three or five degrees of freedom are needed for the group delay approximation over the passband.

The aim of this paper is to apply and extend the approach proposed by Unbehauen [9, 10] for design of IIR filters with constant group delay and Chebyshev stopband attenuation. Some new relations and an examination of the problem will be given. Several design examples based on MatLab are obtained with some new and interesting conclusions in the output results. A generalization for all types of filters (lowpass, highpass, bandpass, and bandstop) will be shown.

II. CONSTANT GROUP DELAY SPECIFICATION

Let us consider an all-pole transfer function of degree m [7, 9] :

$$H(z)=\frac{z^m}{a_0+a_1 z+\ldots+a_m z^m} \quad . \tag{1}$$

The coefficients a_μ are to be adjusted in such a way that the phase [9] :

$$\theta(\omega)=arg\left[1/H(e^{j\omega T})\right], \quad 0\le\omega\le\pi/T \tag{2}$$

approaches a prescribed function $\theta_0(\omega)$ as close as possible. Here ω is the variable radian frequency and T the time period of the digital filter.

Our task is to solve the problem of approximating the ideal phase function :

$$\theta_0(\omega)=\tau.\omega\ , \quad \tau=\text{const}>0 \tag{3}$$

that corresponds to a constant group delay. Replacing ω with W=cos($\omega T/2$), it was proved [9], that the following statement holds :

$$\frac{k.\prod_{\mu=1}^{m}(W-X_{2\mu-1})}{\prod_{\mu=1}^{m-1}(W-X_{2\mu})}=\sqrt{1-W^2}.\cot\left[\theta_0\left(\frac{2}{T}\cos^{-1}W\right)+m.\cos^{-1}W\right] \tag{4}$$

with $-1\le W\le 1$. The left-hand function is approximating odd rational function $f_m(W)$ with parameters $X_{2\mu-1}$, $X_{2\mu}$, and k introduced in [9]. From $f_m(W)$ the unknown transfer function $H(z)$ can be calculated directly.

Inserting the prescription (3) into (4) and using :

$$\omega=\frac{2}{T}\cos^{-1}W=\frac{2}{T}\tan^{-1}\frac{\sqrt{1-W^2}}{W} \tag{5}$$

we obtain the following function :

$$f(W)=\sqrt{1-W^2}.\cot\left[q.\tan^{-1}\frac{\sqrt{1-W^2}}{W}\right] \tag{6}$$

with parameter $q=2\tau/T+m$ which must be approximated by the function $f_m(W)$. According to Perron [5] $f(W)$ can be represented by a continued fraction and its m-th convergent is :

$$f_m(W)=\frac{W}{q}\left[1+\frac{(q^2-1)\left.\frac{W^2-1}{W^2}\right|}{|3}+\ldots+\frac{\left\{q^2-(m-1)^2\right\}\left.\frac{W^2-1}{W^2}\right|}{|(2m-1)}\right]. \tag{7}$$

To algorithmize (7) we need a more suitable presentation. At first, let us rewrite $f_m(W)$ as :

$$f_m(W)=\frac{W}{q}\left[s_0+\frac{r_1|}{|s_1}+\ldots+\frac{r_n|}{|s_n}\right]=\frac{W.R_{m-1}}{q.S_{m-1}}\ , \tag{8}$$

where the polynomials R_γ, S_γ can be calculated using the following recurrence relation [5] :

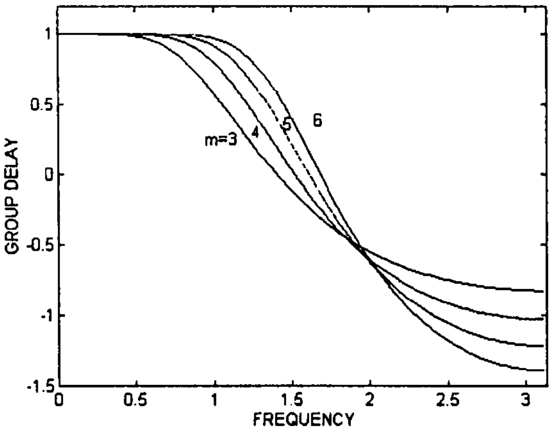


Fig. 1 Dependence of the group delay from degree m for a lowpass filter with $\tau/T=1$

$$\left| \begin{array}{ll} R_\gamma = s_\gamma . R_{\gamma-1} + r_\gamma . R_{\gamma-2} \; , & R_{-1}=1, \; R_0 = s_0 \\ S_\gamma = s_\gamma . S_{\gamma-1} + r_\gamma . S_{\gamma-2} \; , & S_{-1}=0, \; S_0 = 1 \; . \end{array} \right. \qquad (9)$$

Some closed-form expressions obtained by (9) are shown in Table 1. In consequence of the theory of continued fractions [5], the phase function $\theta(\omega)$ of the resulting $H(z)$ approaches the prescription (3) at ω=0 in the maximally flat sense. Furthermore, it can be graphically determined that increasing the order m, the group delay becomes more flat in the passband of the filter (see Fig. 1).

III. CHEBYSHEV STOPBAND ATTENUATION

In this section our task is to summarize and extend the approach given in [9] with application of different squared-magnitude functions $Q(\zeta)$ [10]. We show that this idea works not only for lowpass but also for highpass, bandpass, and bandstop filters. For this purpose, the numerator of an all-pole transfer function will be substituted by a mirror-image polynomial which does not affect the group delay and guarantees equiripple magnitude in a prescribed stopband $\omega_c \le \omega \le \pi / T$.

Starting with the function :

$$Q(z) = H(z)H(z^{-1}) = \frac{1}{(a_0 + a_1 z + \ldots + a_m z^m)(a_0 + a_1 z^{-1} + \ldots + a_m z^{-m})}$$

after transformation $\overline{\mathrm{w}} = (z + z^{-1})/2$ we obtain :

$$Q(\overline{\mathrm{w}}) = \frac{1}{B_0 + B_1 \overline{\mathrm{w}} + \ldots + B_m \overline{\mathrm{w}}^m} \; , \qquad (10)$$

where $Q(e^{j\omega T}) = \left| H(e^{j\omega T}) \right|^2$. Then, two transformations of the frequency are applied consecutively : $\overline{\mathrm{w}} = \alpha.\mathrm{w} + \beta$ and $\mathrm{w} = (\zeta + \zeta^{-1})/2$. Parameters α and β are selected so that the interval in which the squared-magnitude function must approximate a constant value in the equiripple sense, is stretched into the range $-1 \le \mathrm{w} \le 1$. Our investigations showed that we can apply the following squared-magnitude functions $Q(\zeta)$ [10] to approximate a given lowpass, highpass, and bandstop magnitude :

$$\text{- Case 2a : } \quad Q(\zeta) = \frac{(1+\varepsilon) P_s^2(\zeta) - P_a^2(\zeta)}{P_s^2(\zeta) - P_a^2(\zeta)} \qquad (11)$$

$$\text{- Case 2b : } \quad Q(\zeta) = \frac{P_s^2(\zeta) - (1+\varepsilon) P_a^2(\zeta)}{P_s^2(\zeta) - P_a^2(\zeta)} \; , \qquad (12)$$

where ε (<1) is a positive constant corresponding to the stopband ripple. Graphically these two cases are shown in Fig.2. Values of the parameters α and β for lowpass and highpass filters are given in Table 2. The constant 1 was subtracted from the original form of Q as we seek for Chebyshev stopband attenuation. All necessary mappings of the frequency for lowpass, highpass, and bandstop filters are presented in Fig.3 a, b, and c, respectively. For bandstop filters we use Case 2aα with :

$$\alpha = \overline{\mathrm{w}}_D / 2 \; , \; \beta = \overline{\mathrm{w}}_S / 2 \; ,$$

$$\mathrm{w}_l = -(2 + \overline{\mathrm{w}}_S) / \overline{\mathrm{w}}_D \; , \; \mathrm{w}_h = (2 - \overline{\mathrm{w}}_S) / \overline{\mathrm{w}}_D \; ,$$

where : $\overline{\mathrm{w}}_S = \overline{\mathrm{w}}_{C1} + \overline{\mathrm{w}}_{C2}$, $\overline{\mathrm{w}}_D = \overline{\mathrm{w}}_{C1} - \overline{\mathrm{w}}_{C2}$.

As an alternative approach to that given above, we can propose the following method. At first : design of lowpass prototype filter and then using of proper linear mapping. For lowpass to highpass transformation we are able to apply $z \to -z$, lowpass to bandpass : $z \to -z^2$, lowpass to bandstop : $z \to z^2$. For bandstop and bandpass filter the resulting order is doubled. As these transformations are linear, constant group delay property is maintained.

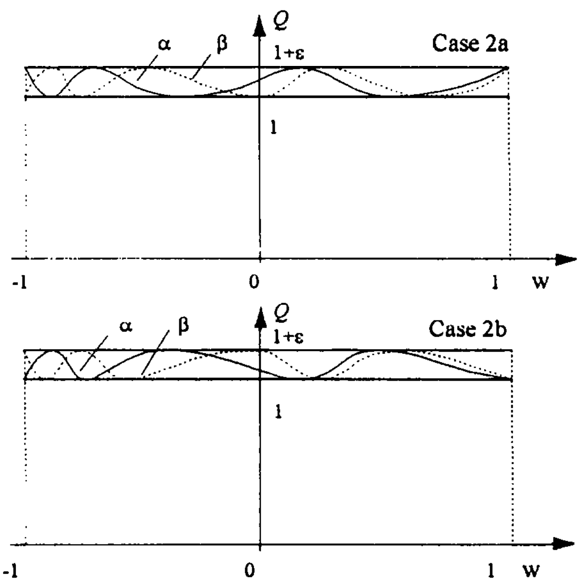


Fig. 2 Squared-magnitude function Q with equiripple behavior in the range [-1,1]

IV. DESIGN EXAMPLES

To demonstrate our approach, several design examples were produced. The graphical results shown in Fig.4 and Fig.5 correspond to three highpass filters obtained with different values of ε and cutoff frequency ω_c . We can see some improvement in stopband attenuation when ε has minimum value (ε=0.05). Also, the amplitude of the all-pole response can be improved with adding of extra ripples for the cases with numerator degree greater than that of the denominator. We must note that in all these cases [9], the denominator should be multiplied by z^r where the value of r ensures an absence of a pole at $z=\infty$.

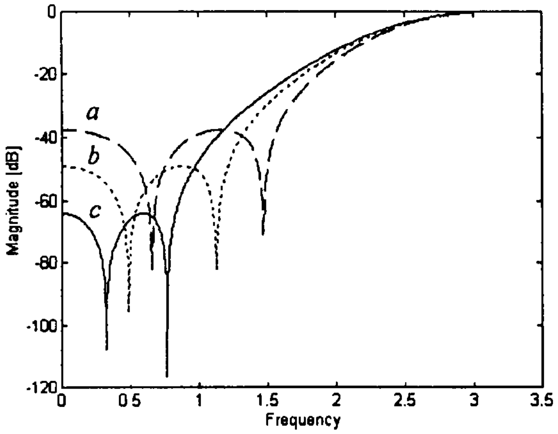


Fig. 4. **Case 2aα** amplitude response with m=3 and numerator degree 4 : *a*) ε=0.1 and $\omega_c=\pi/2$; *b*) ε=0.07 and $\omega_c=\pi/2.6$; *c*) ε=0.05 and $\omega_c=\pi/3.8$.

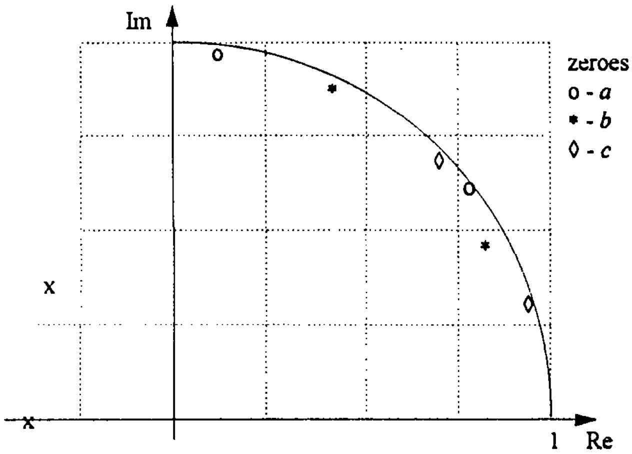


Fig. 5. Pole-zero location of the filters from Fig.4. (poles 'x' are equal for all these cases)

Fig.6 presents the magnitude and group delay for bandpass and bandstop filters obtained by the discussed approach. The transfer function for the bandpass filter is :

$$H(z)=\frac{25.826.(-1+1.323.z^2-1.323.z^4+z^6)}{24+144.z^2+336.z^4+336.z^6}\ .$$

If we want to achieve more flat group delay in the passband of the filter we need a greater degree m (according to results from Section II).

CONCLUSIONS

The proposed design method is intended towards simultaneously amplitude and group delay approximation of IIR digital filters. We use two different squared-magnitude functions to obtain equiripple magnitude behavior in the stopband. A continued fraction expansion approach is applied to achieve constant group delay in the passband. Our method is unified (with application for all types of filters) and gives non-iterative final solution. It could be used as an alternative approach to other methods based on optimization techniques. Our investigations showed that the described method can be successfully extended for the design of filters with equiripple passband.

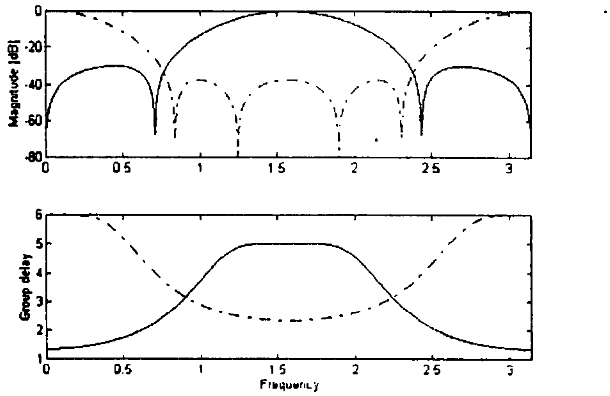


Fig.6 Magnitude and group delay for a bandpass (——) and bandstop filter (-·-·-·)

REFERENCES

[1] A.G. Deczky, "Recursive Digital Filters Having Equiripple Group Delay", *IEEE Trans.Circuits Syst.*, vol. CAS-21, pp.131-134, 1974.

[2] A.Fettweis, "A Simple Design of Maximally Flat Delay Digital Filters", *IEEE Trans. Audio Electroacoust.*, vol.20, pp.112-114, 1972.

[3] S.S.Lawson and A.Wicks, "Design of Efficient Digital Filters Satisfying Arbitrary Loss and Delay Specifications", *IEE Proc.-G*, vol.139, pp.611-620, 1992.

[4] G.A.Maria and M.M.Fahmy, "A New Design Technique for Recursive Digital Filters", *IEEE Trans. Circuits Syst.*, vol.CAS-23, pp.323-325, 1976.

[5] O.Perron, "Die Lehre von den Kettenbrüchen", vol.II, Stuttgart : Teubner, 1957.

[6] T.Saramaki, Y.Neuvo, and T.Saarinen, "Equal Ripple Amplitude and Maximally Flat Delay Digital Filters", in *Proc. IEEE ICASSP*, pp.236-239, 1981.

[7] V.S.Stojanovic and A.D.Micic, "Multiple-Pole Transfer Function with Equiripple Group Delay and Magnitude for Recursive Filter Design", *AEÜ*, vol.47, pp.114-118, 1993.

[8] J.P.Thiran,"Recursive Digital Filters with Maximally Flat Group Delay", *IEEE Trans. Circuit Theory*, vol.18, pp.659-664, 1971.

[9] R.Unbehauen, "Recursive Digital Low-Pass Filters with Predetermined Phase or Group Delay and Chebyshev Stopband Attenuation", *IEEE Trans.Circuits Syst.*, vol.CAS-28, pp.905-912, Sept.1981.

[10] R.Unbehauen, "IIR Digital Filters with Optimum Magnitude Frequency Response", *ntzArchiv*, Bd.7, pp.25-31, 1985.

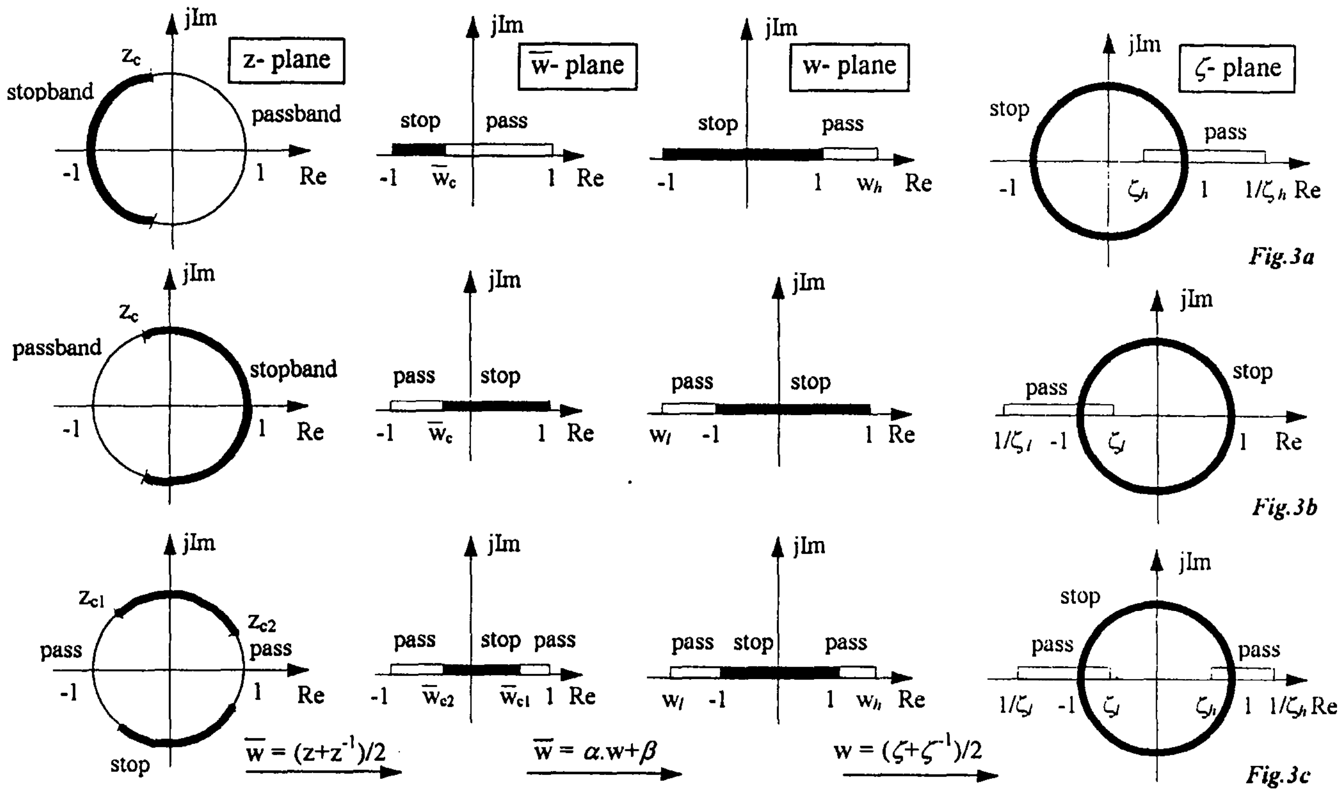


Fig.3a
Fig.3b
Fig.3c

Degree	R_{m-1}	S_{m-1}
$m=3$	$6(q^2+1)-(6q^2-9).W^{-2}$	$11+q^2-(q^2-4).W^{-2}$
$m=4$	$q^4+35q^2+24-(2q^4+25q^2-72).W^{-2}+$ $+(q^4-10q^2+9).W^{-4}$	$10q^2+50-(10q^2-55).W^{-2}$
$m=5$	$15(q^4+15q^2+8)-30(q^4+q^2-20).W^{-2}+$ $+15(q^4-13q^2+15).W^{-4}$	$q^4+85q^2+274-(2q^4+65q^2-$ $-607).W^{-2}+(q^4-20q^2+64).W^{-4}$
$m=6$	$q^6+175q^4+1624q^2+720-3(q^6+105q^4-$ $-406q^2-1800).W^{-2}+3(q^6+35q^4-861q^2+$ $+1350).W^{-4}-(q^6-35q^4+259q^2-225).W^{-6}$	$21(q^4+35q^2+84)-42(q^4+5q^2-$ $-156).W^{-2}+21(q^4-25q^2+99).W^{-4}$

Table 1

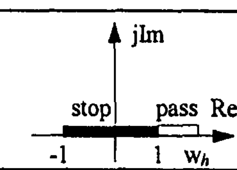
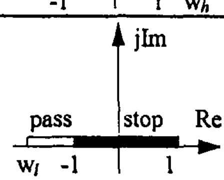
Type	$Q(\zeta)-1$	w - plane	$w_h, \zeta_h, w_l, \zeta_l$	α, β
Lowpass Case 2aβ	$\frac{\varepsilon.P_s^2(\zeta)}{P_s^2(\zeta)-P_a^2(\zeta)}$		$w_h=\frac{3-\overline{w}_c}{1+\overline{w}_c}>1$	$\alpha=\frac{\overline{w}_c+1}{2}>0$
Lowpass Case 2aα			$\zeta_h=w_h+\sqrt{w_h^2-1}$	$\beta=\frac{\overline{w}_c-1}{2}<0$
Highpass Case 2bβ	$\frac{(-\varepsilon).P_a^2(\zeta)}{P_s^2(\zeta)-P_a^2(\zeta)}$		$w_l=\frac{-3-\overline{w}_c}{1-\overline{w}_c}<-1$	$\alpha=\frac{1-\overline{w}_c}{2}>0$
Highpass Case 2aα	$\frac{\varepsilon.P_s^2(\zeta)}{P_s^2(\zeta)-P_a^2(\zeta)}$		$\zeta_l=w_l+\sqrt{w_l^2-1}$	$\beta=\frac{1+\overline{w}_c}{2}>0$

Table 2